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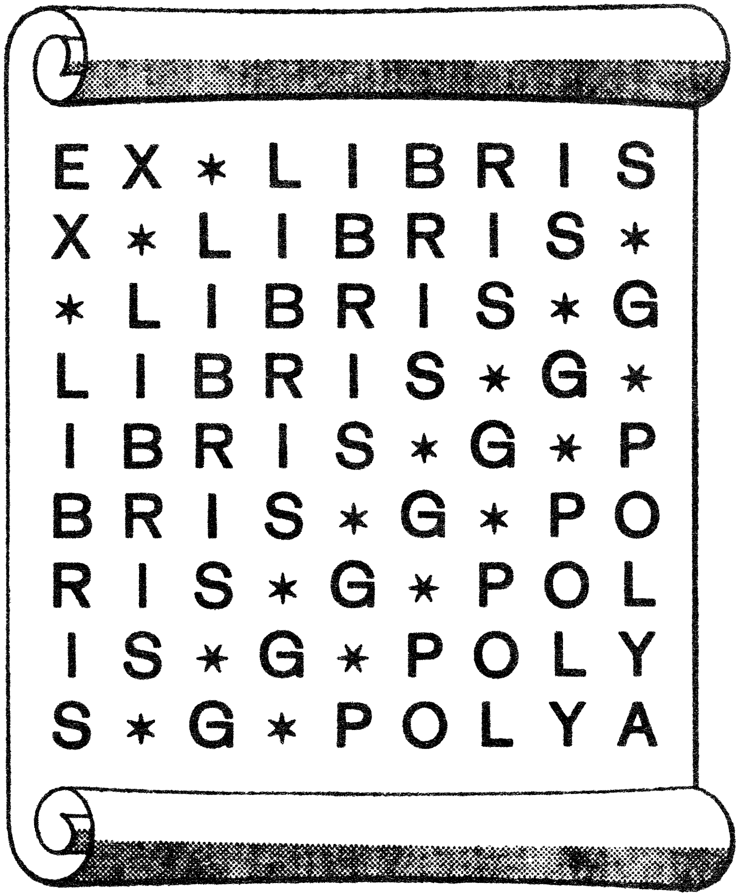

Vol. 60 No. 5
December 1987

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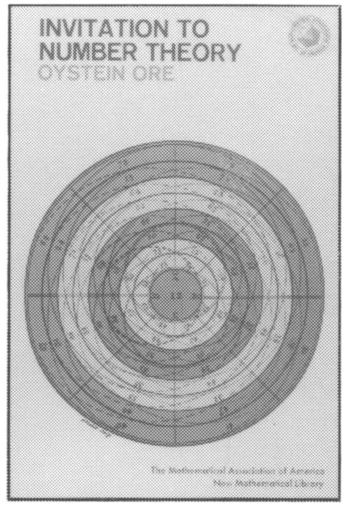

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MATHEMATICS MAGAZINE

Volume 60, Number 5, December, 1987
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ARTICLES

PROBLEMS

ACKNOWLEDGEMENTS

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COVER: G. Pólya's bookplate from his own design.

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The aim of Mathematics Magazine is to provide lively and appealing mathematical exposition. This is not a research journal and, in general, the terse style appropriate for such a journal (lemma-theorem-proof-corollary) is not appropriate for an article for the *Magazine*. Articles should include examples, applications, historical background, and illustrations, where appropriate. They should be attractive and accessible to undergraduates and would, ideally, be helpful in supplementing undergraduate courses or in stimulating student investigations. Articles on pedagogy alone, unaccompanied by interesting mathematics, are not suitable. Neither are articles consisting mainly of computer programs unless these are essential to the presentation of some good mathematics. Manuscripts on history are especially welcome, as are those showing relationships between various branches of mathematics and between mathematics and other disciplines.

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Introduction

When I was asked to edit *Mathematics Magazine* a few years back, I arranged to visit my predecessor, Doris Schattschneider, to find out how one goes about editing a journal. One suggestion she had was that a special issue of the *Magazine* be put together for Pólya's hundredth birthday. Since I had been a student of Pólya's at Stanford many years earlier, I did not need much persuasion and I soon started on preliminary planning.

In 1985 Pólya died at the age of 97, so this issue in honor of Pólya can no longer be a birthday celebration, but it can still mark the centenary of his birth, December 13, 1887. Authors from whom I solicited contributions for this issue came forth with enthusiasm, to contribute recollections, some words on Pólya's mathematics or his influence on teaching, some mathematics, whatever.

Readers who want more detail about the events of Pólya's life or who would like to read at greater depth appraisals of his influence on the various areas of mathematics in which he worked are directed to the obituary in the *Bulletin of the London Mathematical Society* (to appear). Because Pólya worked in so many areas in mathematics—real and complex analysis, probability, combinatorics, number theory, mathematical physics, among others—the list of contributors to that obituary was unusually high. Specialists had to be found in each of the areas in which he worked.

Since Pólya contributed in so many ways in so many fields, it is impossible in the small amount of space available to give a survey of his achievements. The articles only give a sample. But his name has been used repeatedly to describe many of the mathematical ideas that he developed. To list mathematical ideas named for a person gives only a superficial indication of that person's influence on the subject, but I thought in this case it might be of interest to list at least some of the theorems, methods, and mathematical concepts named for Pólya or for Pólya and others. The list is surely not complete and in many cases I suspect a more accessible reference could be given. But the following is a list of named theorems and such with the name "Pólya" in the title, along with a reference in each case for those who might be interested in learning what the theorem is about. Of course, such a list cannot include the ways in which Pólya contributed to the language of mathematics—the first introduction of "random walk" and "central limit theorem", for example.

Hardy-Littlewood-Pólya-Everitt Inequality

Christer Bennewitz, A general version of the Hardy-Littlewood-Pólya-Everitt inequality, *Proc. Roy. Soc. Edinburgh Sect. A*, 97 (1984) 9–20.

Laguerre-Pólya Functions

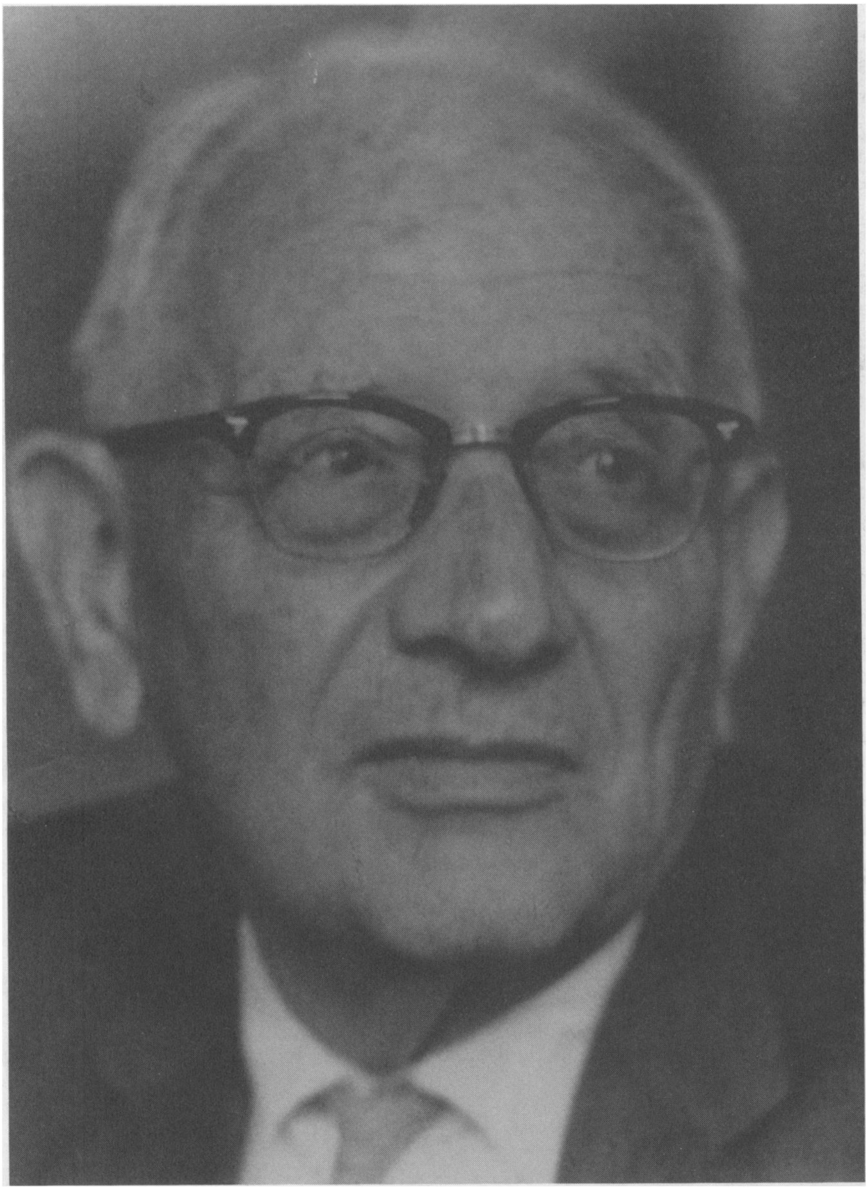
(See Pólya-Schur Functions.)

Lindwart-Pólya Theorem

S. Hellerstein and J. Korevaar, Limits of entire functions whose growth and zeros are restricted, *Duke Math. J.*, 30 (1963) 221–228.

Markov-Pólya Distributions

K. G. Janardan and B. Raja Rao, Characterization of generalized Markov-Pólya and generalized Pólya-Eggenberger distributions, *Comm. Statist. A—Theory Methods*, 11 (1982) 2113–2124.



George Pólya, Göttingen, c. 1953

Nevanlinna-Pólya Theorem
Hiroshi Haruki, A generalization of the Nevanlinna-Pólya theorem in analytic function theory, *Math. Notae*, 29 (1981) 29–35.

Payne-Pólya-Weinberger Inequality
G. Pólya, *Collected Papers*, Vol. 3, MIT Press, 1984, p. 519.

Plancherel-Pólya-Nikol'skij Inequality
Bernd Stöckert, Ungleichungen vom Plancherel-Pólya-Nikol'skij-Typ in gewichteten L_p^{Ω}-Räumen mit gemischten Normen, *Math. Nachr.*, 86 (1978) 19–32.

Pólya's Algorithm
Andras Kroo, On the convergence of Pólya's algorithm, *J. Approx. Theory*, 30 (1980) 139–148.

Pólya-Bernstein Theorem
Maurice Blambert and Rajagopalan Parvatham, Compléments à des théorèmes de S. Mandelbrot et de Pólya-Bernstein, *C. R. Acad. Sci. Paris Ser. A-B*, 290 (1980) A457–A460.

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Khira Lameche, Extension d'un théorème de Pólya-Cantor à des séries rationnelles en variables non commutatives, Séminaire Delange-Pisot-Pointou, 1970/71.

Pólya-Carlson Theorem
Ralph Boas, *Entire Functions*, Academic Press, 1954, p. 178.

Pólya Characteristic Functions
A. I. Il'inskii, The arithmetic of Pólya characteristic functions, *Mat. Zametki*, 21 (1977) 717–725.

Pólya Conditions
G. Pólya, *Collected Papers*, Vol. 3, MIT Press, 1984, p. 494.

Pólya Conjecture
Harold M. Stark, *An Introduction to Number Theory*, Markham, 1970, p. 7.

Pólya's Counting Theorem
(See Pólya Enumeration Theorem.)

Pólya Criterion
G. Pólya, *Collected Papers*, Vol. 4, MIT Press, 1984, p. 613.

Pólya Density
Robert M. Berk, Some monotonicity properties of symmetric Pólya densities and their exponential families, *Z. Wahrsch. Verw. Gebiete*, 42 (1978) 303–307.

Pólya Distribution
William Feller, *An Introduction to Probability Theory and Its Applications*, 3rd ed., Vol. 1, Wiley, 1950, p. 142.

Pólya-Eggenberger Distributions
(See Markov-Pólya Distributions.)

Pólya Enumeration Theorem
Alan Tucker, Pólya's enumeration formula by example, this MAGAZINE, 47 (1974) 248–256.

Pólya Frequency Functions
I. J. Schoenberg, On Pólya frequency functions I. The totally positive functions and their Laplace transforms, *J. Analyse Math.*, 1 (1951) 331–374.

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T. Kovari, On the gap theorem of Pólya, *J. London Math. Soc.*, 34 (1959) 185–194

Pólya-Lundberg Process

Dietmar Pfeifer, A note on the occurrence times of a Pólya-Lundberg process, *Adv. in Appl. Probab.*, 15 (1983) 886.

Pólya-Macintyre Representation Theory

Paul Malliarin and L. A. Rubel, On small entire functions of exponential type with given zeros, *Bull. Soc. Math. France*, 89 (1961) 175–206.

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A. A. Aizikovich, The Pólya-Mammana factorization of a linear matrix difference operator. I, *Differencial-nye Uravnenija*, 14 (1978) 328–337, 388–389.

Pólya Matrices

G. M. Peterson and Anne C. Baker, On a theorem of Pólya (II), *J. London Math. Soc.*, 39 (1964) 745–752.

Pólya Means

L. A. Rubel, Maximal means and Tauberian theorems, *Pacific J. Math.*, 10 (1960) 997–1007.

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Achim Clausing, Pólya operators. I. Total positivity, *Math. Ann.*, 267 (1984) 37–59.

Pólya's Orchard Problem

Thomas Tracy Allen, Pólya's orchard problem, *Amer. Math. Monthly*, 93 (1986) 98–104.

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Carlos R. Handy, Pólya-Padé resonance reconstruction and singular perturbation theory, *Nonlinear Analysis*, 10 (1986) 391–401.

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W. K. Hayman, *Meromorphic Functions*, Oxford, 1964, p. 101.

Pólya Point Process

Ed. Waymire and Vijay K. Gupta, An analysis of the Pólya point process, *Adv. in Appl. Probab.*, 15 (1983) 39–53.

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Pólya Representations

Ralph Boas, *Entire Functions*, Academic Press, 1954, p. 74.

Pólya-Saint Venant Theorem

G. Pólya, *Collected Papers*, Vol. 3, MIT Press, 1984, p. 500–502.

Pólya-Schiffer Inequality

G. Pólya, *Collected Papers*, Vol. 3, MIT Press, 1984, p. 519.

Pólya-Schoenberg Conjecture

St. Ruscheweyh and T. Sheil-Small, Hadamard products of Schlicht functions and the Pólya-Schoenberg conjecture, *Comment. Math. Helv.*, 48 (1973) 119–135.

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I. J. Schoenberg, On totally positive functions, Laplace integrals and entire functions of the Laguerre-Pólya-Schur type, *Proc. Nat. Acad. Sci. U.S.A.*, 33 (1947) 11–17.

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David W. Walkup, Pólya sequences, binomial convolution and the union of random sets, *J. Appl. Probab.*, 13 (1976) 76–85.

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Christophe Reutenauer, On Pólya series in noncommuting variables, *Fundamentals of Computation Theory*, Berlin/Wendisch-Rietz, 1979.

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J. Steinig, The real zeros of Struve's function, *SIAM J. Math. Anal.*, 1 (1970) 365–375.

Pólya-Szegő Composition Formula
Jacob Burbea, Total positivity of certain reproducing kernels, *Pacific J. Math.*, 67 (1976) 101–130.

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Motosaburo Masuyama, A refinement of the Pólya-Szegő inequality and a refined upper bound of CV, *TRU Math.*, 21 (1985) 201–205.

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G. Pólya, *Collected Papers*, Vol. 1, MIT Press, 1974, p. 771.

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William Feller, *An Introduction to Probability Theory and Its Applications*, 3rd ed., Vol. 1, Wiley, 1950, p. 120.

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J. H. Jordan, Character Sums in $Z(i)/(p)$, *Proc. London Math. Soc.* (3), 17 (1967) 1–10.

Pólya *W*-Property
Miseal Zedek, Cayley's decomposition and Pólya's *W*-property of ordinary differential equations, *Israel J. Math.*, 3 (1965) 81–86.

Pólya-Weinstein Inequality
G. Pólya, *Collected Papers*, Vol. 3, MIT Press, 1984, p. 508.

Seeing a list of names tells something of the extent of the man's mathematical influence. Unfortunately, it tells nothing of the influence of his personality, his warmth, his humor, his sense of excitement about mathematics and the teaching of mathematics. I hope that this issue of the MAGAZINE will convey some of those qualities to the reader.

—*Editor*

Zeitgeist

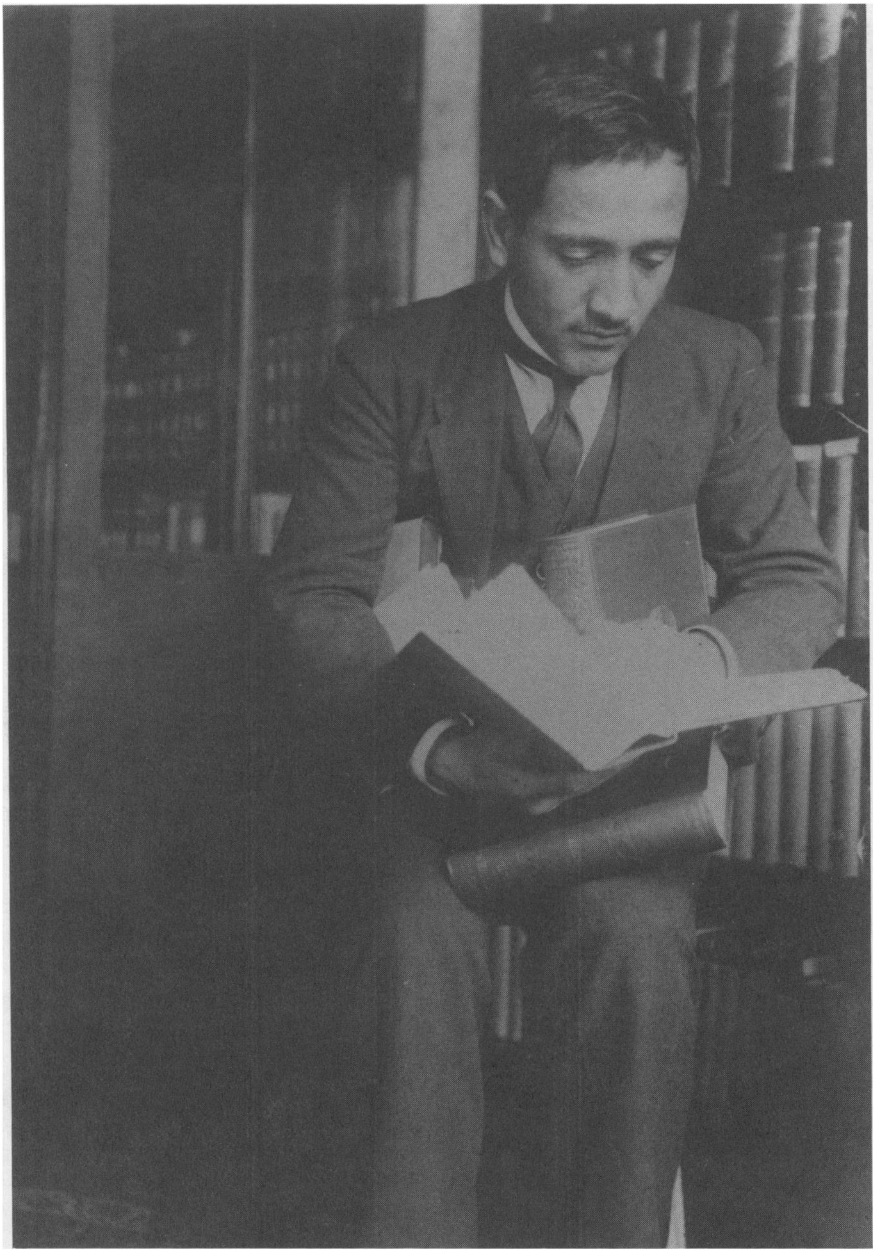
When Oswald Spengler wrote his book[1]
About the West's decline, he took
As measures of a culture's heart
Its mathematics and its art.

Thus Doric columns correlate
With the Apollonian fate,
And thus the concept "function" can
Define the form of Faustian Man.

What would he say if he could see
Our interest in catastrophe,
Or how from order chaos rises,
Full of bifurcate surprises?

J. D. MEMORY

[1] O. Spengler, *The Decline of the West*, A. A. Knopf, Inc., 1928.



Pólya during his student days in Budapest

A Conversation with George Pólya

AGNES ARVAI WIESCHENBERG
John Jay College of Criminal Justice-CUNY
New York, NY 10019

This profile is based on interviews conducted by the author on February 3–4, 1983, and May 8, 1985, in the home of George Pólya in Palo Alto, California.

There is an unexplainable attraction between people sharing common roots, no matter how remote in the past. I first met Pólya (a fellow countryman) after a short correspondence and a telephone call, yet I approached him with the kind of excitement one feels visiting an old friend one has not seen in many years. It is impossible not to get emotional upon meeting such a giant of a human being, one who has done so much for mathematics and the teaching of it for almost a whole century. Our conversation took us back to his teens and the gymnasium (college preparatory secondary school) he attended. He remembered the teachers who impressed him and influenced him to select a career in mathematics.

George Pólya was born in Budapest on December 13, 1887. His father was a lawyer and his brother, Jenő Sándor Pólya, who was 11 years older, a world-renowned physician.

Young Pólya was not particularly interested in mathematics. He attended the Markó Street Gymnasium in Budapest. Some of the outstanding teachers whom he remembered at this school were his Latin, Hungarian, and geography teachers, all of whom were members of the Hungarian Academy. Out of the three mathematics teachers he had in the gymnasium, Pólya remembered "two were despicable and one was good." This may explain the fact that Pólya was not yet interested in pursuing a career in mathematics.

Pólya participated in two self-study groups, one in mathematics and physics and the other in literature. He received a prize in the literary group for translating Heine into Hungarian, an achievement about which he was still embarrassed because he misunderstood and mistranslated a word. Pólya remembered himself as being a good problem solver at this age, but not particularly interested in this endeavor. In the gymnasium he never secured a ranking of first in his class but ranked from second to fourth, which he maintained with ease. The Eötvös Competition in mathematics, which entering college students were encouraged to take, did not excite him. He remembered going to the test center but not handing in his paper.*

Because of his academic achievements Pólya received a tuition exemption from the University of Budapest (1905). Upon the insistence of his mother, Pólya spent his first semester in law school, which he found terribly boring. Through his readings of Darwin (*The Descent of Man*) he

*The Eötvös Competition in Hungary was the first nationwide mathematics competition in the world. It was administered each fall for the graduates of the gymnasium (college preparatory secondary schools) on a voluntary basis. Among the winners one can recognize mathematicians and physicists who are known around the world. Today the competition is known as the Kürschák Competition.

became interested in biology, but his brother insisted that "there is no money in biology"; so Pólya dropped the idea and turned to languages and literature. He received a Latin and Hungarian teaching certificate that he never used. At this point Pólya turned to philosophy. Professor Alexander, his philosophy professor, advised him to take courses in physics and mathematics as part of his studies in philosophy. This decision had far-reaching consequences for his career. His physics course was taught by Professor Loránd Eötvös, a great physicist, who was also known for his devotion to science and teaching. According to Pólya, Eötvös was the best teacher at the University. Pólya's fascination with physics can be attributed to the memorable lectures of Professor Eötvös.

Another person who exercised great influence on Pólya's development was Lipót Fejér, who worked with distinction on Fourier series and was a well-known mathematician with a magnanimous nature. Pólya recalled: "Almost everybody of my age group was attracted to mathematics by Fejér." Indeed, besides Pólya, his students included Marcel Riesz, Otto Szász, Jenő Egerváry, Mihály Fekete, Ferenc Lukács, Gábor Szegő, Simon Szidon, later Paul Csillag and Tibor Radó, and even later Paul Erdős and Paul Turán. Fejér would sit in a Budapest cafe with his students and solve interesting problems in mathematics and tell stories about mathematicians he had known. A whole culture developed around this man. Although his lectures were considered the experience of a lifetime, his influence outside the classroom was even more significant because it was around him that the first meaningful mathematical "school" developed in Hungary.

Pólya's decision to become a mathematician came indirectly, and the traces of his attraction to philosophy and physics remain evident in his work. Although physics was his first love he admitted he chose mathematics because "I thought I am not good enough for physics and I am too good for philosophy. Mathematics is in between."

Pólya was awarded the Ph.D. in 1912 at the University of Budapest after spending a year at the University of Vienna (1910–11). His Ph.D. was in mathematics with a minor in physics and chemistry. The thesis topic was "Some questions of the calculus of probability, and some definite integrals associated with it."

After completing his doctorate, Pólya did postdoctoral work at the University of Göttingen (1912–13) and at the University of Paris (1914).

Pólya got his first teaching position at the Swiss Federal Institute of Technology in Zürich, in 1914. The Hungarian Army rejected him when he tried to sign up for duty, due to a childhood soccer injury that had resulted in blood poisoning and an operation on his leg. When the war situation got desperate, the Hungarian Army called him to report from Switzerland, but Pólya decided not to return to Hungary because he was convinced, through the readings of mathematician-philosopher Bertrand Russell, that war was wrong. Because of this decision he could not visit his homeland for 30 years. Pólya married a Swiss woman, Stella V. Weber, in 1918. Stella, strikingly beautiful and charming, was a devoted wife to George through 67 years of marriage. The Pólyas had no children.

Like so many Europeans who were helpless and horrified by the activities of Hitler, the Pólyas left for the United States in 1940. For two years Pólya had a position as visiting professor at Brown University before settling down in Palo Alto, a town he loved, where he received an appointment at Stanford University.

Pólya spoke four languages fluently (equally badly he explained): Hungarian, French, German, and English. He read and understood a few more.

Among the numerous books that he wrote, he seemed to have been proudest of *How to Solve It* (1945), which has sold almost one million copies and was translated into 17 languages. As the teaching of problem-solving becomes the direction in mathematics education in the eighties, his book will enjoy further success in the years to come. His other great work in mathematics education, *Mathematics and Plausible Reasoning* (1954), translated into six languages, reflected his knowledge and interest in philosophy. *Mathematical Discovery* (1962), translated into eight languages, and the *Stanford Mathematics Problem Book* (1974), co-authored with J. Kilpatrick, also enjoy great interest among mathematics educators.

His works in mathematics are also numerous: *Problems and Theorems in Analysis* (1925), co-authored with Gábor Szegő, a colleague and friend at Stanford; *Inequalities* (1934), co-authored with Hardy and Littlewood; and *Isoperimetric Inequalities in Mathematical Physics* (1951), co-authored with Szegő, reflecting his interest in physics. *Complex Variables*, co-authored with G. E. Latta (1974), is another of his well-known publications. His collected papers fill four volumes.

George Pólya was prominent and respected around the world for his work in mathematics education. Although he only occasionally visited Hungary, his influence on mathematics education there is still most significant today. His teaching style—sometimes referred to as the "Pólya style"—has been captured on tape by the Mathematical Association of America and is presented as an example to students at teacher's colleges.

Among his long list of honors were: honorary member of the Hungarian Academy, the London Mathematical Society, the Swiss Mathematical Society, the Society for Industrial and Applied Mathematics; member of the National Academy of Sciences (USA) and the American Academy of Arts and Sciences; and corresponding member of the Académie des Sciences (Paris).

Pólya received an award for distinguished service to mathematics from the Mathematical Association of America in 1963, and he won the "blue ribbon" in the Educational Film Library Association's 10th Annual Film Festival in 1968, in the category of "Mathematics and Physics," for his lecture series called "Let Us Teach Guessing." This film was produced by the Mathematical Association of America.

When asked which mathematician's work influenced him most, he answered Euler, because "Euler did something that no other great mathematician of his stature did. He explained how he found his results, and I was deeply interested in that. It has to do with my interest in problem-solving."

Among his teachers, Eötvös and Fejér at the University of Budapest and Hurwitz in Zürich impressed him the most. Among colleagues, it was with Gábor Szegő that he had the longest and closest collaboration. Szegő, who was also born and raised in Hungary, had been a student at the University of Budapest and emerged from the group of mathematicians studying under Fejér. While chairman of the Mathematics Department at Stanford University, Szegő initiated a competitive mathematics examination (1946) for high school seniors in California, modelled on the Eötvös competition. (Szegő himself had won the Eötvös competition in 1912.) The purpose of this examination was to stimulate interest in mathematics among high school students as well as teachers, and to identify talented students. The winner was awarded a one-year scholarship to Stanford. The program was discontinued in 1965. Pólya, who worked closely on this project with Szegő, said in explanation, "I told them it was a mistake to discontinue the program."

When I asked Pólya what accounted for the development of so many outstanding Hungarian mathematicians from the period around the turn of the century, he gave the following reasons:

Generally speaking, mathematics is the cheapest science. Unlike physics or chemistry, it does not require any expensive equipment. All one needs for mathematics is a pencil and paper. (Hungary never enjoyed the status of a wealthy country.)

Among specific reasons:

1. *The Mathematics Journal for Secondary Schools*. This publication was put out by the gymnasium (secondary school) teachers for their students. The journal stimulated interest in mathematics and prepared students for the Eötvös Competition.

2. The Eötvös Competition. The competition created interest and attracted young people to the study of mathematics.

3. Professor Fejér. He alone was responsible for attracting many young people to mathematics, not only through his formal lectures but through his informal meetings and discussions with his students.

Right up to his final illness and death at the age of 97, George Pólya was compassionate, wise, and witty. In a way he was still an active member of the mathematical community. As recently as 1978 he was teaching a course at Stanford and giving lectures around the country.

Although his eyesight failed in his last years, Pólya read and answered his correspondence personally. His complaints about "aging" were more frequent. He often reminded his visitors that he was almost 100 years old. His sense of humor, however, helped him through difficult times. "My mathematical interest is not dead yet," he explained, "but I seldom feel fit to do mathematics."

While we briefly discussed computers and what they can do for the teaching of mathematics, Pólya would immediately admit that computers are something he knew nothing about. Although —and I take this as the greatest compliment—he said "I am almost 100 years old, too old to learn computers but if I would live in New York, I would listen to your computer classes."

In a telephone conversation Paul Erdős, the itinerant mathematician, told me that he had promised Pólya a big celebration on his 100th birthday. To this suggestion Pólya replied "maybe 100, but not more."

Pólya died on September 7, 1985 in Palo Alto.

George Pólya (1887–1985)

(Remarks by M. M. Schiffer at a memorial service for Pólya at Stanford University, October 30, 1985.

We have come here together to honor the memory of George Pólya, a great mathematician, a superb teacher and writer, a respected colleague, and a beloved friend to many of us. His scientific activity spans a large period of over 75 years and has affected significantly many branches of mathematics; to mention only a few: complex analysis, analytic number theory, theory of probability, combinatorics, geometry and applied mathematics. The list of his collaborators includes some of the greatest names of a golden age of mathematics and attests also to his great gift for friendship and stimulating cooperation. The great French mathematician and his good friend, Jacques Hadamard, wrote a book on the *Psychology of Mathematical Invention* and there he classified all mathematicians into two types: those who think that they discover theorems and ideas which are outside of them and wait to be found; the second type who think that they make mathematics by establishing a logical structure at will and then look for the consequences. The first type is often enthusiastic about art, especially music; the second type is more rigorous but somewhat plodding. Both types are essential and necessary to our science. George Pólya was definitely of the first type; the driving force in his research was the search for beauty and the joy of discovery. You can judge the intellectual character of a person by his interests in extracurricular matters. George was very attracted by philosophy and literature. There are writers whom one respects and even admires, and there are always writers whom one loves. George's favorites of the second kind were Voltaire, Anatole France, and Heinrich Heine. They typify in literature the mentality of Pólya in mathematics: wit, sparkle and brilliance. He was proud that he and Heine had the same birthday, December 13th. He founded, by the way, the 13th birthday club which included friends and colleagues who are born on the 13th day of some month. This was a funny affair, but it gave rise to many cordial social celebrations. His main interest and passion in life was, of course, mathematics. Many of you know his joke about how he came to study mathematics: "I was not smart enough to become a physicist, and too smart to be a philosopher, so I chose mathematics." This is, of course, not true and just shows his love for a clever epigram. He was attracted to mathematics by teachers and friends like L. Fejér, Frédéric and Marcel Riesz, and many others in the flourishing Budapest group of mathematicians. Once he tasted the joy of discovery, it never let him go. He wished to teach people to experience his feeling which made him such an outstanding lecturer and writer. In his last years his eyesight became very bad and he could read only with great difficulty. So he bought himself a reading machine where the printed page appears on something like a television screen, very magnified and illuminated. So he started reading again with great enthusiasm. Very much of his reading material was his own papers and books. He told me: "You know, when I was younger, I was extremely clever." This was not conceit or pride. He relived the moments of discovery and enjoyed them again. You can understand his deep interest in heuristics, the art and love of discovery, to which he devoted a large part of his time and interest. Heuristics has many pioneers like Descartes, Leibnitz, Helmholtz, Poincaré, and Hadamard. But Pólya is today its most outstanding representative. This interest was a tremendous benefit for his students who enjoyed, of course, his masterful teaching, based on the principle that the student should be helped to rediscover for himself the matter he was taught. Pólya has written or co-authored over 250 papers in many languages. His collected works have recently been published in four big volumes by the M.I.T. Press. His books were still more influential. He wrote with Hardy and Littlewood *Inequalities*, with his lifelong friend, Gabor Szegő, *Theorems and Problems of Analysis* and *Isoperimetric Inequalities*, with Latta, *Complex Variables*. By himself he wrote *How to Solve It*, *Mathematics and Plausible Reasoning*, *Mathematical Discovery*, *Mathematical Methods in Science*. He was a brilliant expositor; each book is a classic, still in print and in use today. Last September I was at a meeting at the University of Maryland devoted to function theory, attended by many experts from various countries and universities, where I received a telephone call from Stanford, informing me that George had passed away. When I told this to my colleagues at the meeting, everyone told me how much Pólya's books (*Problems and Theorems* in particular) have affected their career and taste in mathematics.

I remember a conversation with George when we talked about the great Danish physicist, Niels Bohr, who created completely new insights in theoretical physics and found great difficulties in conveying these ideas to his colleagues. He was accused of being hard to understand. He defended himself by quoting the German poet Schiller: "In der Breite liegt die Klarheit, in der Tiefe wohnt die Wahrheit." (In the breadth lies the clarity, in the depth dwells the verity.) George knew and liked this quotation but insisted that in mathematics truth and clearness could be achieved together and his life's work confirms that. He went early in his life to Zürich, attracted there by Adolf Hurwitz to whom he was closest and whose collected works he later edited. I think it was Hermann Weyl who called him the great aphorist in mathematics. Every paper of Hurwitz was a gem: crystal clear, beautiful, and deep. No wonder George was attracted by him, for this was also his style. His views and ideas had also great influence outside of mathematics. Books in psychology of thought refer to him often. Let me mention Arthur Koestler, who was very interested in heuristics. He wrote *The Sleepwalkers*, which follows the thought processes of the founders of modern astronomy and *The Act of Creation*, which studies the psychology of invention. He came to Stanford once specifically to discuss this matter with Pólya. The Dutch engineer and artist Escher consulted with Pólya about his work on symmetry groups, which play such an important role in Escher's graphic work. Pólya is often called a classic mathematician, this is true in the sense that his papers and books have a classical beauty and elegance, not in the sense that he is not modern. True, he knew and loved the work of Euler and the grand masters of the 18th and 19th century, but he has gone far beyond them. He was careful to preserve the old achievements and somewhat cautious about trends and fads in mathematics. Typical in this respect is his famous bet with his friend and colleague in Zürich, Hermann Weyl. At the beginning of this century, in a true crisis in the foundations of mathematics, the Dutch mathematician Brouwer made some deep investigations in the foundations of mathematics and raised doubts about the logic of some mathematical reasoning, in particular the "indirect proof." He and Weyl founded a school in mathematics, the so-called intuitionism. Weyl bet with Pólya that fifty years from their meeting mathematics would be completely revamped from this point of view. They laid it down in a document to be opened in 50 years and Pólya won the bet, hands down. He was indeed conservative; in the 40 years of his residence in Palo Alto he never learned

Pólya in his study in Palo Alto, c. 1947

how to drive. But there is an interesting counterpart to this. For a quarter of a century, he did not like to travel by air, and since he traveled extensively in this country and overseas, this was a tremendous handicap. In 1964 he was in Zürich and we had a conference on function theory in London. When he received the invitation to participate and time for surface travel was too short, his love for mathematics overcame his aversion and he flew. On arrival in London he told me with great surprise that air travel was pleasant indeed, and so, after his 75th year he became an enthusiastic air traveler and crossed the continent and the Atlantic several times. By the way, he was of course invited to give a talk at the conference and he talked about "How I did not prove the Riemann conjecture." It was an unprepared but very interesting talk, very typical of him.

His great achievements brought him many honors and distinctions. I cannot enumerate his honorary doctor degrees. I mention only that he was a member of the American Academy of Arts and Sciences, of the National Academy of Sciences, of the Hungarian Academy of Sciences, and corresponding member of the Académie des Sciences of France. This Académie has a very rigid establishment and one can trace the history of each chair from its beginning. It gave George great pleasure to trace his membership to its first occupant, Isaac Newton. In our Mathematics Library there is only one portrait on display, that of Pólya. The only building called after a mathematician at Stanford is Pólya Hall, which houses the Department of Numerical Analysis.

George Pólya had a long, creative, and happy life. Filled with enthusiasm for his science, the warmth of many friendships, his taste for art, he enjoyed his life to the fullest. The most important factor in all this was his happy marriage to his wife Stella, which lasted for 67 years. She gave him support, the cultured and comfortable home, and the affection which is so necessary in an active and creative life. We all are sad to have lost George, but Stella's bereavement is much greater. A good and gentle man has left us. But his work will continue to affect our science, and we will always remember him with love and admiration.

Selected Topics from Pólya's Work in Complex Analysis

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1. Singular points. You learned in a calculus course how to find the Taylor series of a function, but even in a course in complex analysis you may not have encountered the problem of recovering the properties of a function from its Taylor series. Since the coefficients of the series determine the function, it must be true that every property of the function is implicit in the sequence of coefficients—but it is quite a different problem to make the sequence tell you whether or not the function has a particular property.

For a simple example, consider a power series that converges in the unit disk. One of the important questions to ask about a function is: Where are its singular points? Some functions can be continued analytically to points outside the disk of convergence; for example, $\Sigma_{n=0}^{\infty} z^n$ defines a function ($1/(1-z)$, of course) which has a singular point at $z = 1$ and no other singular points. How could we discover this by looking at the sequence $\{1, 1, \dots\}$ of coefficients? On the other hand, $\Sigma_{n=0}^{\infty} z^{2^n}$ defines a function that has the unit circle as a natural boundary: every point on the circle is singular. Here the sequence of coefficients is $\{0,1,1,0,1,0,0,0,1,0,0,0,0,0,0,0,1,0,\dots\}$, and the noticeable feature of this sequence is that the nonzero terms are very scarce: there are many "gaps" in the sequence of coefficients. It is not very hard to prove that the circle of convergence is a natural boundary for this function, but it is harder to prove Hadamard's theorem that every series with its nonzero coefficients separated by sufficiently long gaps has the same property. A more precise statement is that if the series is $\Sigma a_k z^{n_k}$ and $n_{k+1}/n_k \geq \lambda > 1$ then the sum of the series cannot be continued beyond the circle of convergence. In the example with which we began, $\lambda = 2$.

Hadamard's theorem is rather far from the ultimate truth. We need to be able to say in a precise way how sparse the nonzero coefficients in a series are. One measure of sparseness is the number $D = \liminf_{k \to \infty} k/n_k$. For $n_k = 2^k$ we have $D = 0$, and in fact $k/n_k \to 0$ quite fast. We would have $D = 0$ if, for instance, $n_k = k^2$ or $k \log k$. Can we still say something about the singular points of the function in these cases? The positive answer to this question is Fabry's gap theorem, which says that $D = 0$ guarantees that every point of the circle of convergence is singular. This is a difficult theorem: even after almost a century there is no really simple proof.

2. Gap theorems. At this point we are prepared to appreciate some of Pólya's major contributions. He proved, in the first place, that Fabry's theorem is best possible **[157]***: the only sequences $\{n_k\}$ for which we can say that *every* series $\Sigma a_k z^{n_k}$, with finite radius of convergence, has its circle of convergence as a natural boundary are those for which $D = 0$.

The signs of the coefficients also have something to say about the singular points of a function. For example, if all $a_k \geq 0$ and $\Sigma a_k z^k$ has circle of convergence $|z| = R$, then the point $z = R$ is a singular point (Vivanti's theorem). Pólya proved **[26, 113]** that, as long as the circle of convergence is finite, it can always be made into a natural boundary by changing the signs of the coefficients; in fact, $\Sigma \pm a_n z^n$ has $|z| = R$ as a natural boundary for "most" sequences of $\pm$ signs. **[35]** (However, in most cases we do not know which sequences of signs will work.)

As you have probably realized, mathematicians are never satisfied: they always ask whether a theorem can be generalized. Suppose, as a first generalization, that we weaken the density condition $D = 0$ to $D = \delta$, where $0 < \delta < 1$. For a unit circle of convergence, Fabry showed (in

*References in this form are to Pólya's bibliography as given in each volume of the *Collected Papers*, MIT Press, 1974, 1984.

Pólya's formulation) that there is then a singular point on every arc of length greater than $(1-\delta)2\pi$ of the circle.

Another generalization is to consider series of the form $\Sigma a_k z^{n_k}$, where the n_k are not necessarily integers. There is ambiguity about which branch of the powers to take; we can avoid this by writing the series as $\Sigma a_k e^{n_k \log z}$, and then replacing $\log z$ by w, so that we have $\Sigma a_k e^{n_k w}$. A series of this form is called a Dirichlet series, and its region of convergence is a half plane rather than a disk. Thus we are led to ask when the sum of a Dirichlet series can be continued across the bounding line of the half plane of convergence. The theorems that result are analogous to, but much harder than, the corresponding theorems for power series. The density has to be defined in a different way, and the interesting case is that in which the exponents are not too close together, more precisely $n_{k+1} - n_k \geq c > 0$. Then if the series converges for $\operatorname{Re} z > 0$ but not for $\operatorname{Re} z < 0$, and $\{n_k\}$ has density D, every interval of length greater than $2\pi D$ on the imaginary axis contains a singular point [**105, 106**]. The proof of this theorem was one of Pólya's most spectacular accomplishments.

3. Other special coefficients. It is always interesting to have a theorem whose hypothesis is simple and whose conclusion seems disproportionately far-reaching. Pólya was interested in many theorems of this kind about power series. Suppose, for example, that we look at a power series whose radius of convergence is 1 and whose coefficients are integers (as often happens). Pólya conjectured, and proved in special cases, that there are only two possibilities: the sum of the series is either a rational function or a function with the unit circle as a natural boundary. (The full theorem was proved by Carlson; later Pólya proved stronger results [**29, 69, 72, 110**]). Pólya studied other similar classes of coefficients, for example the case of only finitely many different coefficients; but I will leave these for you to look up for yourself; the introduction to Vol. I of the *Collected Papers* is a good place to start.

Here is another "all or nothing" result of Pólya's [**24, 29, 57**]. There are entire functions that take integral values at the positive integers, for example, all polynomials with integral coefficients. Can you think of any others?... You might come up with n^z, where n is a positive integer ($n > 1$). These functions have much more complicated behavior near the point ∞ than polynomials have: they have ∞ as an essential singular point, whereas polynomials have poles at ∞. How close to a polynomial can an entire function be if it has integral values at the positive integers? We measure the "size" of an entire function by its order and type; roughly, the order is the infimum of exponents k such that $|f(z)| \leq Ae^{B|z|^k}$ for some A and B; the type is the infimum of admissible B's. Thus $n^z = e^{z \log n}$ is of order 1 and type $\log n$. Pólya discovered that 2^z is the smallest transcendental entire function with integral values at the positive integers [**24**]. This has led to many generalizations by Pólya [**29, 57**] and many other authors.

4. The Pólya representation. Pólya systematized his methods for dealing with problems about entire functions in a long paper [**113, 137**]; parts of it are summarized in [**76, 113a**]. This paper includes important material on the structure of real sequences and on convex sets; the central result is a representation theorem for entire functions which are of exponential type, that is, of order 1 and finite type. Such functions (examples are e^{cz}, $\sin cz$, and so on) have Laplace transforms, whose analytic continuations are analytic outside a convex set, which is called the conjugate indicator diagram D; the original entire function f has the representation

$$f(z) = \frac{1}{2\pi i}\int_C \varphi(w)\, e^{zw}\, dw,$$

where $\varphi(z)$ is the analytic continuation of the Laplace transform of f, and C is a curve that surrounds the conjugate indicator diagram. This representation can be used in many ways. One straightforward way to apply it is to expand φ in some kind of series and integrate term by term to get a series expansion for f.

5. Zeros. As you know, we cannot have explicit algebraic formulas for the roots of general polynomial equations of degree greater than 4, much less of equations $f(z) = 0$ where f is a more general analytic function. On the other hand, there are many reasons for wanting to find the zeros of functions: to tell an engineer whether a control system is stable; to locate critical points (zeros of derivatives); or simply in order to know whether we can divide one function by another. The precise location of the zeros of the Riemann zeta function is not completely known to this day, but the distribution of these zeros governs the distribution of the prime numbers. The prime numbers are no longer merely playthings of mathematicians, but are used, among other places, in practical problems of cryptography. According to a remark of Norbert Wiener's, the distribution of the zeros of the zeta function (hence of the primes) also governs the astrophysical problem of analyzing the radiation from a distant distribution of black bodies at various temperatures.

Nowadays there are fast computer programs for locating the zeros of polynomials with numerical coefficients, but when the coefficients are not explicit numbers we are generally grateful for any information about where the zeros of particular kinds of polynomials are. The classical result is the Gauss-Lucas theorem, which says that the zeros of the derivative of a polynomial are in the convex cover of the set of zeros of the original polynomial. The derivative is only a special case of a transformation of a polynomial by doing something to the coefficients. What sort of transformations will work? Pólya's answer depends on the following, apparently unrelated problem.

One of the less intuitive properties of analytic functions is that a convergent sequence may lose zeros in the limit. For example, the partial sums of the Maclaurin series of e^z converge very nicely to e^z, but (being polynomials) they acquire more and more zeros as their order increases, whereas their limit has no zeros. This sort of phenomenon raises the question of what functions can be limits of polynomials whose zeros are required to be in a particular set, say in a half line, a line, or a half plane. When Pólya began to study problems of this kind, a certain amount was already known, but nobody else had realized how far it was possible to go.

Pólya's fundamental results [**16, 17, 18, 20, 22**] were that if an analytic function f is approximable in a neighborhood of zero by polynomials with only real zeros or only real positive zeros, then (unless $f(z) \equiv 0$) the approximation will be uniform on every compact set, and f has to be an entire function of rather slow growth and a very special form. These functions (now known as Pólya-Schur or Laguerre-Pólya functions) turned out to be of fundamental importance in many problems, and in particular in the question of what transformations of a polynomial do to its zeros.

6. Zeros of derived functions. We can ask (as a generalization of the Gauss-Lucas theorem) when $L(D)f$ has its zeros in the convex cover of the set of zeros of f, where L is defined by a power series. If $L(z) = \sum_{n=0}^{\infty} \alpha_n z^n$, then $L(D)f$ means the result of applying the formal operator $\sum \alpha_n D^n$ to f, where D stands for differentiation. If K is a line or a half line, a necessary and sufficient condition [Pólya and Schur, **20**] is that $\sum (\alpha_k / k!) z^k$ is one of the Pólya-Schur functions. If K is the lower half plane, then L itself must be a Pólya-Schur function [**95**].

7. Repeated differentiation. Once we think of looking at the zeros of derivatives, it is natural to think about differentiating more than once. Pólya devoted a great deal of attention to problems of where the zeros of a function go under repeated differentiation. One of his most elegant results was for functions that are meromorphic in the plane. Here the zeros condense on the polygon whose sides are equidistant from the two nearest poles, as if the poles acted as force centers that repel the zeros [**67**].

For entire functions the problem is much more difficult. For example, Pólya believed that if f is entire and real on the real axis (hence symmetric in the sense that $f(\bar{z}) = \overline{f(z)}$), the nonreal zeros migrate toward the real axis if f is of order less than 2, but away if the order is greater than 2. This belief has now been largely confirmed, in part by Pólya's own work. The detailed results are too elaborate to present here; if you are interested, see [**167**] for the situation as of 1943;

Pólya's *Collected Papers*, vol. 2, up to 1974; and the obituary in the *Bulletin of the London Mathematical Society* for references to more recent work.

8. Zeros near a point. We know that an analytic function all of whose derivatives are zero at a point must vanish identically. Suppose, however, that the zeros of the derivatives are concentrated near a point; then what happens? This is an old problem; in one especially nice version, we suppose that f is an entire function of order 1, type τ, that is, of exponential type τ. Suppose that for every n we have $f^{(n)}(z_n) = 0$ with $|z_n| \le 1$. It may seem plausible that having τ small enough might force $f(z) \equiv 0$. Takenaka proved in 1932 that $\tau = \log 2 = 0.693\ldots$ is small enough, and the example $\varphi(z) = \sin(\pi z/4) - \cos(\pi z/4)$ shows that $\pi/4 = 0.785\ldots$ is not small enough. What is the critical value of τ? This question remained open for many years. The zeros of the derivatives of $\varphi(z)$ are at ± 1; this suggested that a function ψ with zeros at the vertices of an equilateral triangle might do better. Pólya never published anything on this problem, but he proved (in correspondence) that $\log 2$ is too small and that $\pi/4$ is too large. The best possible value of τ has been approximated numerically, but the details have not yet been published.

Pólya was able to estimate the smallest zero of ψ by using Graeffe's method, which was originally used for polynomials but is adaptable to functions defined by power series. For many years the method was considered to involve too much computation for practical use, but with modern computers it has become useful; see P. Henrici, *Applied and Computational Complex Analysis*, vol. 3. Since the method is not very well known, I should like to show you how it works in simple cases; it is particularly well adapted to approximating the zero closest to 0. Let

$$\psi(z) = \sum_{n=0}^{\infty} a_n z^n, \qquad a_0 = 1.$$

If ω is a kth root of unity and we form

$$\psi(z)\psi(\omega z) \cdots \psi(\omega^{k-1} z),$$

we get a power series that starts

$$1 + b_1 z^k + b_2 z^{2k} + \cdots.$$

If $|z|$ is small $z = (-1/b_k)$ will approximate the smallest root of $\psi(z) = 0$.

I have presented only some of the most easily described highlights of Pólya's work in complex analysis; the work as a whole fills two large volumes of his collected papers. I have said nothing, for example, about his contributions to conformal mapping. If you find this kind of mathematics interesting, you should go directly to the *Collected Papers*. Much of what I have mentioned is not easily available anywhere else, and everything that Pólya wrote is well worth reading. I myself spent all my disposable time for two years editing the two volumes on complex analysis, and have never regretted it.

If you cannot solve the proposed problem try to solve first some related problem.

G. Pólya,
How To Solve It,
Princeton, 1945, p. 10

An idea which can be used only once is a trick. If you can use it more than once it becomes a method.

G. Pólya and G. Szegő,
Problems and Theorems in Analysis,
Springer, 1972, vol. 1, p. viii

Pólya's Theorem and Its Progeny

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1. Pólya's Theorem

In 1937 there appeared a paper [9] that was to have a profound influence on the future of combinatorial analysis. It was one of George Pólya's most famous papers, and in it Pólya presented to the combinatorial world a powerful theorem which reduced to a matter of routine the solution of a wide range of problems. Moreover, this theorem contained within it the potential for growth and generalization in many directions. In this note I give a brief account of Pólya's Theorem and of various developments to which it gave rise.

Pólya's Theorem relates to the enumeration of mathematical objects called 'configurations,' which, in an abstract sense, can be defined as mappings from a set D to a set R. The elements of R are called 'figures' and each figure has a 'content'—usually a non-negative integer. A more concrete way of picturing a configuration is to take the set D to be a set of 'locations,' or 'boxes.' Then a mapping $f\colon D \to R$ is equivalent to the process of putting one figure in each box, allowing the possibility of the same figure being put in two or more boxes. Each configuration, that is, each mapping, has a content, defined as the sum of the contents of the figures making up the configuration. A standard combinatorial problem is then to determine the number of configurations having given content.

This problem was made more general by the introduction of a group G of permutations of the elements of D. Two configurations are said to be equivalent if one can be obtained from the other by permuting the elements of D by some permutation belonging to G. This corresponds to the physical situation in which the 'boxes' can be shuffled in some way which results in a configuration regarded as being essentially the same configuration as the original.

An example will make this clear. Suppose the configuration is a necklace of eight beads, each bead being either black or white. The necklace has no clasp, and so we would regard the necklace as remaining the same if we rotate it or flip it over. Define the content of a figure, or of a configuration, to be the number of black beads. We then have a set of eight boxes—the set D—into which we put one of two figures, a black bead or a white bead, having content one and zero respectively. This is then a typical Pólya-type problem. We ask for the number of inequivalent configurations having given content, n say; that is, the number of necklaces having exactly n black beads (and hence $8-n$ white beads). Naturally we must know and use information about the group G, which in this problem is the dihedral group D_8 of rotations and flips of the necklace. This information is provided by the 'cycle-index' of the group, defined as follows. Take each element, g of G and express it as a product of disjoint cycles (it is well known that every permutation can be so expressed, uniquely except for the order of the cycles). If g gives j_i cycles of length i, we associate with g the monomial $s_1^{j_1}s_2^{j_2}\cdots s_n^{j_n}$ (where n is the degree of G, i.e., the size of the set D). This monomial is called the 'cycle-type' of G. The cycle-index is then defined to be the average of these cycle-types, and by collecting up terms having the same cycle-type we can express it in the following form

$$Z(G; s_1, s_2, \ldots) = \frac{1}{|G|}\sum A_{(j)} s_1^{j_1}s_2^{j_2}\cdots s_n^{j_n},$$

where the expression on the left is the standard notation for the cycle-index, and on the right, $|G|$ is the order of G, $A_{(j)}$ is the number of permutations whose cycle-type is $s_1^{j_1}s_2^{j_2}\cdots s_n^{j_n}$, and the summation is over all partitions $1^{j_1}2^{j_2}\cdots n^{j_n}$. For the dihedral group D_8 the cycle-index is easily verified to be

$$Z(D_8)=\frac{1}{16}\left(s_1^8+4s_1^2s_2^3+5s_2^4+2s_4^2+4s_8\right).$$

Information about the figures and their contents is given in the form of a counting series (otherwise known as a generating function) of the form

$$f(x)=a_0+a_1x+a_2x^2+\cdots,$$

where a_r is the number of figures which have content r. This is known as the figure counting series. For our necklace problem this reduces to $f(x)=1+x$ since there are only two figures. The answer to the problem is given as another counting series—the configuration counting series—defined as

$$F(x)=A_0+A_1x+A_2x^2+\cdots,$$

where A_r is the number of configurations of content r. Given the figure counting series and the cycle index of the group in question, the configuration counting series is given by Pólya's Theorem, which we can now state.

Pólya's Theorem. *The configuration counting series is obtained by substituting the figure counting series into the cycle index. By this is meant that, in the cycle index, we replace every occurrence of* s_i *by* $f(x^i)$.

We can see at once how this applies to our necklace problem. Substituting the figure counting series $1+x$ into $Z(D_8)$ we obtain

$$\frac{1}{16}\left[(1+x)^8+4(1+x)^2(1+x^2)^3+5(1+x^2)^4+2(1+x^4)x^2+4(1+x^8)\right],$$

which reduces to

$$1+x+4x^2+5x^3+8x^4+5x^5+4x^6+x^7+x^8.$$

Thus, for example, there are five inequivalent necklaces having three black beads, and eight with equal numbers of black and white beads. The latter are shown in FIGURE 1.

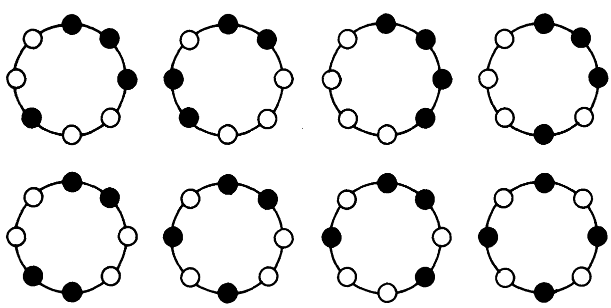


FIGURE 1

2. Applications

The applicability of Pólya's theorem is not confined to such trivial matters as counting necklaces. As an example of a more serious application, we can turn to the subject of enumerating chemical compounds, a topic which makes up a considerable portion of Pólya's paper. In the molecule of an 'alkane' (formerly known as a paraffin) if there are n carbon atoms then there are $2n+2$ hydrogen atoms. Every carbon atom is joined by single bonds to four other atoms; each hydrogen atom is joined to one other atom, in fact a carbon atom. FIGURE 2 shows some typical examples, all for $n=5$.

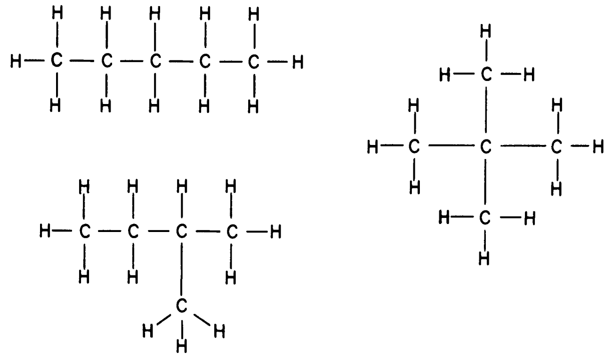


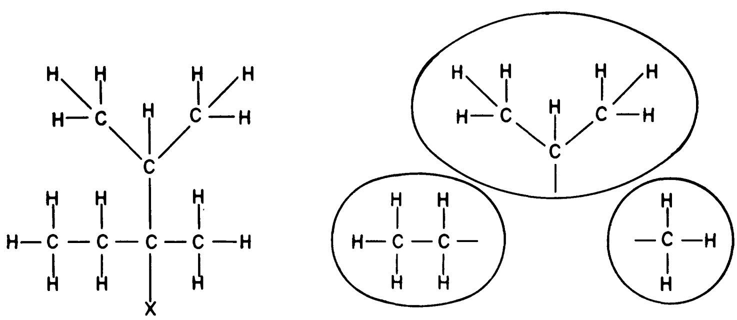
FIGURE 2

Great interest attaches to finding how many such formulae there are for a given number of carbon atoms, and we shall consider one such problem; but to make things easier we shall take the case where one of the hydrogen atoms is replaced by something else—denoted here by X. This gives us a point of attack on the problem. These 'substituted alkanes' are the configurations to be enumerated. An example is shown in FIGURE 3a.

Denote by A_n the number of substituted alkanes having exactly n carbon atoms. We therefore seek the configuration counting series

$$A(x) = A_0 + A_1 x + A_2 x^2 + A_3 x^3 + \cdots .$$

Now at the carbon atom (the 'root' atom) to which the X is attached there are three other bonds, which can be regarded as the 'X-ends' of three substituted alkanes of smaller size, as in FIGURE 3b. We can call these 'side-chains'. Hence in this problem we have three boxes and the set of figures is the set of substituted alkanes itself. If we bear in mind that the three bonds are, so to speak, rigidly attached to the carbon atom in question, we see that the group G is most reasonably taken to be the cyclic group C_3, whose cycle index is



(a) (b)

FIGURE 3

$$Z(C_3) = \frac{1}{3}\left(s_1^3 + 2s_3\right).$$

Applying Pólya's Theorem, and substituting $A(x)$ into this cycle index we get

$$\frac{1}{3}\left[A^3(x) + 2A(x^3)\right].$$

In this configuration counting series the content is the number of carbon atoms in the side chains alone. We must multiply by a further x in order to count the root atom, and we then have the configuration counting series for substituted alkanes, that is, for $A(x)$. Thus we obtain the equation

$$A(x) = \frac{x}{3}\left[A^3(x) + 2A(x^3)\right]$$

from which the values of A_i can be calculated recursively.

This, and many other enumerations of chemical compounds, were given in Pólya's paper in great detail. Much more has been done along these lines by later writers (see, for example, [**14**]).

Another important application of Pólya's theorem is to the enumeration of unlabelled graphs. Pólya was aware of this application, but did not publish anything on it himself. Because it provides a good demonstration of the use of Pólya's Theorem, and because of later developments, it will be briefly described.

We consider graphs on a given fixed number p of vertices and enumerate these according to the number of edges. There are $p(p-1)/2$ pairs of vertices, and each of these is a potential edge. Think of these pairs as the boxes for the problem, and let there be two figures 'non-edge' and 'edge', with contents zero and one, respectively. Then, as with the necklace problem, the figure counting series is $1+x$. If the vertices are permuted by some permutation g belonging to the full symmetric group S_p, then the pairs of vertices will be permuted among themselves. Thus the group S_p of permutations of the vertices induces a group of permutations of the edges, denoted $S_p^{(2)}$. The cycle index of $S_p^{(2)}$ can be found, and we are then ready to apply Pólya's Theorem and obtain the configuration counting series which solves the problem. A fuller description of this process and similar applications to many other problems was given in [**4**], and further details can be found in [**6**].

3. deBruijn's Theorem

The first of several extensions and generalizations of Pólya's Theorem was that of deBruijn [**1**]. His generalization consisted in the introduction of another group, say H, which permutes the figures, in addition to the group G of permutations of the boxes. Two configurations were then regarded as equivalent if one could be obtained from the other by permuting both boxes and figures by appropriate permutations. A good example is provided by the necklace problem already considered. Suppose that, in addition to the previous conditions, we now allow the interchange of the two colours. Then the number of inequivalent necklaces will be reduced. deBruijn's theorem counts the necklaces from this new point of view and shows that the answer is now seven. The reason for this is that the first two necklaces in FIGURE 1, formerly distinct, are now equivalent. The remainder, however, are invariant under this 'complementation' of colours —they are self-complementary. In a similar way the application of deBruijn's theorem to the problem of enumerating graphs leads to the enumeration of self-complementary graphs, that is, graphs that are unchanged by the interchange of the two figures 'edge' and 'non-edge.' For details on this problem see [**12**].

4. Power Group Enumeration

The ties between Pólya's Theorem and deBruijn's Theorem were strengthened in 1966 with the publication by Harary and Palmer of their power group enumeration theorem. Like deBruijn's Theorem this relates to a set of mappings $f\colon D \to R$, with a group G acting on D and another group H acting on R. These mappings are treated in the same way as the boxes in Pólya's Theorem, and under the action of the two groups they are permuted among themselves in a rather complicated manner. Thus the two groups G and H induce a group of permutations of the mappings, denoted H^G and called the 'power group.' The problem of enumerating mappings then reduces to that of finding the cycle index of this group, followed by the application of

Pólya's Theorem. Harary and Palmer gave formulae for computing the cycle index in question, and hence solving problems of the deBruijn type, as well as some more general problems along the same lines. For more information on the power group enumeration theorem see [**5**], [**6**].

5. Superposition

Another theorem that can be used to solve problems similar to those to which Pólya's theorem applies is the superposition theorem found in [**11**]. The scope of this theorem is most easily seen in a graph theoretical setting, in which it relates to the number of non-isomorphic graphs that can be formed by superposing two graphs G_1 and G_2 on one and the same set of vertices. It is understood that edges from the two graphs are distinguished, say by giving them different colours. Thus if G_1 is the graph on six vertices consisting of two triangles, and G_2 is the graph consisting of three disjoint edges, as shown in FIGURE 4, it is easily verified that there are just two ways of superposing these two graphs; they are shown in FIGURE 5, where the edges are distinguished by bold and dotted lines.

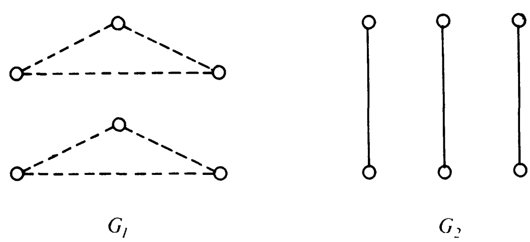


FIGURE 4

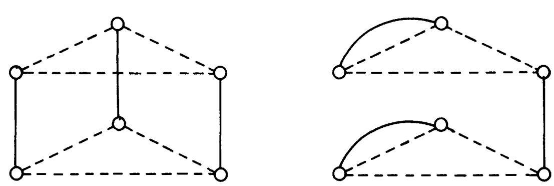


FIGURE 5

It turns out that the required number of superpositions depends only on the cycle indexes of the automorphism groups of G_1 and G_2. In fact, if these cycle indexes are

$$\frac{1}{|G_1|}\sum_{(j)} A_{(j)} s_1^{j_1} s_2^{j_2} \cdots s_p^{j_p}$$

and

$$\frac{1}{|G_2|}\sum_{(j)} B_{(j)} s_1^{j_1} s_2^{j_2} \cdots s_p^{j_p}$$

(with the same notation as before), then the required number is

$$\frac{1}{|G_1||G_2|}\sum_{(j)} A_{(j)} B_{(j)} 1^{j_1} 2^{j_2} \cdots p^{j_p} \cdot j_1! j_2! \cdots j_p!. \tag{1}$$

The connection with Pólya's theorem may not be immediately apparent, but can be illustrated by an example. Suppose we take G_1 to be the cycle of length 8, and G_2 to consist of two complete graphs each on four vertices. These are shown in FIGURE 6. A typical superposition is shown in FIGURE 7. Compare this latter figure with the first diagram in FIGURE 1. There is an obvious connection. In fact there is a one-to-one correspondence between these superpositions and the necklaces in FIGURE 1, with G_1 corresponding to the string of the necklace, while one component of G_2 indicates the positions of the black beads and the other of the white beads. The automorphism group of G_1 is D_8, and that of G_2 is the direct product $S_4 \times S_4$. From the superposition theorem we obtain the number 8 in agreement with what we had before. In this example we distinguished the two components of G_2. If we do not, then we have the analogue of the example used to illustrate deBruijn's Theorem, where black and white could be interchanged. In this case the automorphism group of G_2 is the 'wreath product' $S_2[S_4]$, the group in which allowance is made for the interchange of the two components of G_2. The revised computation then yields the value 7, agreeing with that given by deBruijn's Theorem.

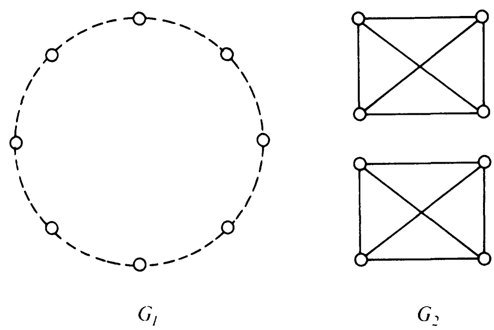


FIGURE 6

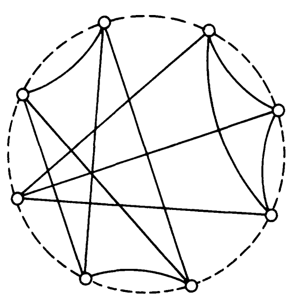


FIGURE 7

Thus the superposition theorem can be used, among many other things, to find individual coefficients in the counting series produced by Pólya's Theorem, deBruijn's Theorem or the power group enumeration theorem, without having to find the whole series. For more information on superposition see [**6**], [**11**].

6. J. H. Redfield

No account of the current topic would be complete without mention of the work of J. H. Redfield. In 1960 Frank Harary startled the combinatorial world by pointing out that this virtually unknown mathematician in a paper [**15**] published in 1927, had anticipated much of the development in combinatorial analysis for the next thirty years, including Pólya's Theorem! It is true that the theorem was not stated with the clarity of Pólya's exposition, nor were its consequences so thoroughly investigated, but it was undeniably there, along with many other results. This remarkable paper contained, for example, a theorem more general than the superposition theorem, namely that if we form the expression

$$\frac{1}{|G_1||G_2|}\sum_{(j)} A_{(j)}B_{(j)}1^{j_1}2^{j_2}\cdots p^{j_p}\cdot j_1!j_2!\cdots j_p!s_1^{j_1}s_2^{j_2}\cdots s_p^{j_p}$$

(in the notation used earlier), then this polynomial is the sum of the automorphism groups of the superposed graphs. From this the expression (1) follows at once from the fact that, by definition, the value of a cycle index when all the s_i are put equal to 1 is always 1.

Redfield's paper had an immediate effect, so to speak (it was, in reality, a very belated effect) on the progress of combinatorial mathematics. It encouraged a point of view that was already becoming manifest, which linked enumeration theory in general and Pólya's Theorem in particular, to the theory of symmetric functions. This was achieved by regarding the s_i as being power sums in some set of indeterminates. In this way every cycle index becomes a symmetric function, and the standard symmetric functions become cycle indexes of particular importance. See **[13]** for further details. Redfield's paper also led enumerators to explore the possibilities of working not with just the numbers of configurations, but with the sums of the cycle indexes of their automorphism groups. As a result of this new approach to the subject, many problems which had previously been considered quite intractable, were thus rendered capable of solution. Among the many graph theoretical enumeration problems which were surmounted in this sort of way were those of counting connected graphs with no vertices of degree 1, unlabelled blocks and acyclic digraphs. Information on these topics can be found in **[3]**, **[16]**, **[17]**.

7. Further Applications

Apart from its numerous applications to purely mathematical problems, Pólya's Theorem has proved useful in many problems arising in the natural sciences and in other disciplines—problems relating to real-life situations. It would be impossible in a comparatively short article to give anything like a complete account of these uses to which Pólya's Theorem and its progeny have been put. At most we can just look at a few selected topics.

Some chemists have been among the most avid users of Pólya's Theorem. This is not altogether surprising since Pólya himself devoted quite a lot of his paper to problems of chemical enumeration. The enumeration of acyclic hydrocarbons (treelike molecules of carbon and hydrogen only) has been carried out to great lengths, as has the enumeration of the similar compounds with fewer hydrogen atoms (having, in consequence, some double or triple bonds between the atoms), and of substitutional derivatives of these compounds. For an overview of this see **[14]**. Other work, starting with a paper by Pólya himself **[8]**, relates to the enumeration of cyclic compounds having a common basic structure. In the search for solutions to problems of this type chemists have been led to formulate further extensions of Pólya's Theorem (see for example **[7]**, **[18]**). Rather less common applications in chemistry are to the study of crystal structure and of nuclear magnetic resonance spectrography.

Pólya's Theorem has also been found to be useful in physics, engineering, telecommunications and other branches of science and technology. There is even an application to music theory! For further information about such applications the interested reader can refer to my companion article to the recently published translation of Pólya's paper **[10]**. This translation should serve to make known to a wider audience the contents of this important milestone in the history of combinatorial mathematics—a brilliant paper by one of the most outstanding mathematicians of our time.

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Euler's Other Proof

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$$
\begin{aligned}
\frac{\pi^2}{6} &= \frac{4}{3}\frac{(\arcsin 1)^2}{2} = \frac{4}{3}\int_0^1 \frac{\arcsin x}{\sqrt{1-x^2}}\,dx \\
&= \frac{4}{3}\int_0^1 \frac{x + \sum_{n=1}^{\infty} \frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots 2n}\frac{x^{2n+1}}{2n+1}}{\sqrt{1-x^2}}\,dx \\
&= \frac{4}{3}\int_0^1 \frac{x}{\sqrt{1-x^2}}\,dx + \frac{4}{3}\sum_{n=1}^{\infty}\frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots 2n(2n+1)}\int_0^1 x^{2n}\frac{x}{\sqrt{1-x^2}}\,dx \\
&= \frac{4}{3} + \frac{4}{3}\sum_{n=1}^{\infty}\frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots 2n(2n+1)}\left[\frac{2n(2n-2)\cdots 2}{(2n+1)(2n-1)\cdots 3}\right] \\
&= \frac{4}{3}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} = \frac{4}{3}\left(\sum_{n=1}^{\infty}\frac{1}{n^2} - \frac{1}{4}\sum_{n=1}^{\infty}\frac{1}{n^2}\right) = \sum_{n=1}^{\infty}\frac{1}{n^2}.
\end{aligned}
$$

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Pólya, Problem Solving, and Education

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Nineteen forty-five was both a banner year and a year of great chaos for problem solving. It was the year that Max Wertheimer's *Productive Thinking*, a classic study of problem solving, first appeared in English. Jacques Hadamard's "Essay on the Psychology of Invention in the Mathematical Field" appeared as well. So did Karl Duncker's monograph *On Problem Solving*. At the same time mathematics instruction in the schools was mostly drill-and-practice, since it was based on the Associationists' "theory of bonds" (see **[16]**). Drill-and-practice was under attack, however. With Wertheimer leading the charge, the Gestaltists claimed that such instruction completely missed the substance of mathematical thinking. The Gestaltists argued that rote instruction was of little value, and that when students memorized without understanding they missed the underlying essence of the mathematics they studied. Unfortunately, the Gestaltists had no theory of learning or instruction, since they believed all the real "action" takes place in the subconscious. Hence they had few practical classroom suggestions. Moreover, they were counterattacked by the behaviorists, who claimed that "mind" and "thinking" were useless constructs, and that all behavior (mathematical and otherwise) could be explained by stimulus-response chains.

It was against that background of psychological and pedagogical confusion that the most important problem-solving work of the time appeared. As indicated in the next section, *How to Solve It* was hardly Pólya's first foray into the world of problem solving. It was, however, an absolutely critical one. *How to Solve It* marked a turning point both for its author and for problem solving. For Pólya it was the first of a series of major volumes on the nature of mathematical thinking, the topic that became the focus of his work in his later years. *How to Solve It* was followed in 1954 by the two volumes of *Mathematics and Plausible Reasoning*, and in 1962 and 1965 by the two volumes of *Mathematical Discovery*. For mathematics education and for the world of problem solving it marked a line of demarcation between two eras, problem solving before and after Pólya. Since then Pólya's influence both on the study of mathematical thinking and on the study of productive thinking in general has been enormous. One major purpose of this note is to trace out the main ideas in Pólya's work. Another is to trace the influence of those ideas on subsequent research into the nature of mathematical thinking.

Pólya and "modern heuristic"

Pólya's interests in the nature of mathematical thinking and in mathematical pedagogy were apparent early in his career. They had already borne significant fruit by 1925, when Pólya and Szegő produced their landmark problem book, *Aufgaben und Lehrsätze aus der Analysis I*. The English version of the book, *Problems and Theorems in Analysis I*, was translated in 1972 and still has the cachet of the original. That book is a good place to start our description of Pólya's work, for its epigraph reads:

What is good education? Giving systematically opportunity to the student to discover things by himself.

It announces the two main pedagogical themes that Pólya was to pursue for more than half a century. The two themes, woven together through Pólya's works, are order (or systematicity) and

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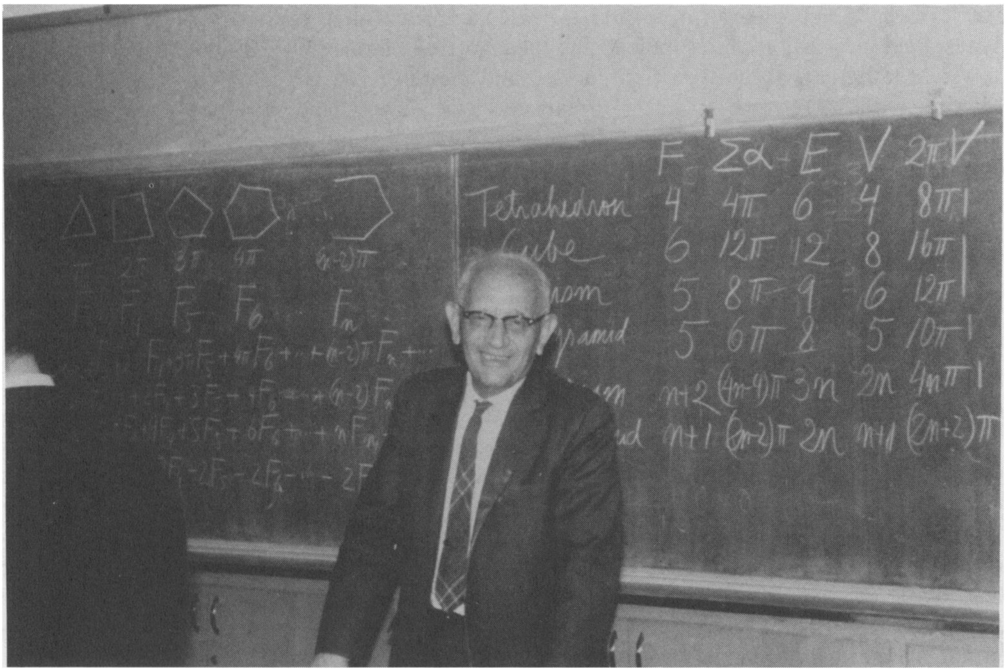
discovery—the systematic discovery of mathematics, and the systematic exposition of a "method" (à la Descartes) for discovery and invention. The two volumes of *Problems and Theorems in Analysis* are primarily concerned with the former. In them Pólya and Szegő seek to reveal the structure of specific mathematical areas through carefully sequenced problems—or as they put it, "to accustom advanced students of mathematics, through systematically arranged problems in some important fields of analysis, to the ways and means of independent thought and research" [**8**, Preface, p. vi]. To this day, anyone who has worked through the problem books has mastered the basics of advanced mathematics, has done some significant problem solving, and is probably ready for serious research. Yet it should be noted that, even then, mathematical structure was not all that concerned Pólya. Although the two volumes of problems are primarily pieces of mathematical exposition, it is clear that when writing them Pólya had "method" on his mind. In the preface to the first (1925) German edition the authors offered a few aphorisms for productive thinking, and the following comments about their use:

> General rules which could prescribe in detail the most useful discipline of thought are not known to us. Even if such rules could be formulated, they could not be very useful...[for] one must have them assimilated into one's flesh and blood and ready for instant use...The independent solving of challenging problems will aid the reader far more than the aphorisms which follow, although as a start these can do him no harm. [**8**, Preface, p. vii.]

Two decades later Pólya returned, at book length, to the same issue. He had embarked on the study of the history of mathematical invention, a pursuit that would come to fruition in the two volumes of *Mathematics and Plausible Reasoning*. Pólya was no longer content with mere aphorisms. He had begun, in the spirit and tradition of Descartes' *Rules for the Direction of the Mind*, an introspective study of *method*—a study of the patterns of productive thinking that enabled him (and would, it was assumed, enable others) to be successful at mathematical problem solving. Perhaps chastened by the failure of Descartes' *Rules* and informed by his historical studies, Pólya took a more modest approach. His was the study of *heuristic*. "The aim of heuristic is to study the methods and rules of discovery and invention... Heuristic, as an adjective, means 'serving to discover.' Heuristic reasoning is reasoning not regarded as final and strict but as provisional and plausible only, whose purpose is to discover the solution of the present problem" [**9**, p. 112–113].

How to Solve It provides the general outline of a problem solving framework and a hint of the details necessary to implement it. The general framework offers a four-phase description of the problem-solving process: understanding the problem, devising a plan, carrying out the plan, looking back. The details are Pólya's heuristic suggestions, or as he calls them, his modern heuristic. Heuristic strategies are rules of thumb for making progress on difficult problems. There are, for example, heuristic strategies for understanding a problem (focusing on the unknown, on the data, drawing a diagram, etc.), for devising a plan (exploiting related problems, analogous problems, working backwards, etc.), and for carrying out and checking a solution. The "short dictionary of heuristic," an alphabetical compendium of techniques and historical notes comprising the bulk of *How to Solve It*, provides some elaboration.

Like the Gestaltists, Pólya was deeply concerned with mathematical structure and used introspection as a primary vehicle for insight. Unlike the Gestaltists, however, Pólya probed deeply into the sources of his insight. As an example illustrating the contrast one may consider Duncker's classic discussion of the *thirteen problem*: "Why are all six-digit numbers of the form abc, abc divisible by 13?" Gestaltist to the core, Duncker argued that the difficulty with the problem disappears when the fact that all such numbers are divisible by 1001 "emerges from the subconscious." Unfortunately, how or why that particular fact might emerge from the subconscious was left unaddressed. In contrast, Pólya's "modern heuristic" was intended as the vehicle for such insights. How could we deal with the thirteen problem, Pólya would ask, so that we might reasonably expect to discover the factor of 1001? Here are three possible ways, exemplifying three of Pólya's heuristic strategies.



Pólya, c. 1965

Explore the conditions of the problem.
We are given the number *abc*, *abc*. What is this number, really? It's a base 10 shorthand for

$$\begin{aligned} &10^5a + 10^4b + 10^3c + 10^2a + 10b + c \\ &= 100100a + 10010b + 1001c \\ &= 1001(100a + 10b + c), \end{aligned}$$

which reveals the hitherto "mysterious" factor of 1001.

Explore a specific case.
Pick a random three-digit number *abc*, and write it down twice. Say we pick 294, to produce 294, 294. What happens when we say this number out loud?

294 thousand and 294

suggests

$$294(1000) + 294 = 294(1001)$$

and we're done.

Look at simple cases and try to find a pattern.
In class, one of my students suggested that we try the simplest 3-digit number, $abc = 001$. This yielded

$$abc, abc = 001001.$$

Her next choice as $abc = 002$, yielding 2002; then 3003, and so on. It was then obvious that each number of the form *abc*, *abc* had a factor of 1001.

Such were the ideas explored, in preliminary fashion, in *How to Solve It*. They were explored at much greater length and depth in the two volumes of *Mathematics and Plausible Reasoning*: *Induction and Analogy in Mathematics* (Vol. 1) and *Patterns of Plausible Inference* (Vol. 2). In those volumes, filled to the brim with historical examples of mathematical insight and invention, Pólya staked his major philosophical and pedagogical claims. We find them summed up as follows: "Mathematical facts are first guessed and then proved, and almost every passage in this book endeavors to show that such is the normal procedure. If the learning of mathematics has anything to do with the discovery of mathematics, the student must be given some opportunity to do problems in which he first guesses and then proves some mathematical fact on an appropriate level." [**10**, p. 160]

On the philosophical plane Pólya was formulating an empiricist view of mathematics, one that treats mathematics as a discipline of discovery on a par with the natural sciences. Despite the fact that it may appear so from the outside, Pólya argued, mathematics is not a purely deductive science that starts with a few axioms and builds up irrevocably from those. "To a philosopher with a somewhat open mind all intelligent acquisition of knowledge should appear sometimes as a guessing game... The result of the mathematician's creative work is demonstrative reasoning, a proof, but the proof is discovered by plausible reasoning, by guessing." [**10**, p. 158] This claim was backed up by means of extensive historical exegeses which included discussions of Euler's heuristic summation of the series $\Sigma\ 1/n^2$; Newton's extrapolations from projectile motion to the orbits of planets; induction in solid geometry, including some of the development of Euler's formula, $V - E + F = 2$ for polyhedra; developments in number theory, physical mathematics, and the theory of probability. Though our topic here is problem solving and not epistemology, it should be noted that these explorations ultimately had philosophical significance as well. Pólya's mathematical views paralleled the scientific views of Karl Popper, who argued that the development of scientific knowledge consists of much more than merely piling up of mountains of objective "facts." Imre Lakatos was a joint student of Pólya and Popper. In *Proofs and Refutations*, one of the most charming mathematics and philosophy books ever written, Lakatos pursued the history of the development of Euler's formula. His historical treatment shows that basic definitions (like the definition of "polyhedron") are changed by mathematicians when they need to, in the light of new empirical evidence—for example, when a theorem everybody knows ought to be true turns out to be false. In that way, mathematics responds to "data" just as empirical science does. Definitions are revised by mathematicians in the same way that theories are revised by physical scientists.

But let us return to pedagogy. In addition to the philosophical subject matter, the two volumes of *Mathematics and Plausible Reasoning* offered a range of difficult problems through which readers could come to grips with the topics Pólya discussed. Pólya wrote that the book was intended for "students desiring to develop their own ability, and for readers curious to learn about plausible reasoning and its not so banal relations to mathematics. The interests of the teacher are not neglected, I hope, but they are met rather indirectly than directly. I hope to fill that gap some day." [**10**, p. 160]

His attempt to fill that gap came a decade later, in the two (1962 and 1965) volumes of *Mathematical Discovery*. As Peter Hilton notes in the foreword to the (1980) combined paperback edition of *Mathematical Discovery*, the book is "eminently suitable for the needs of all concerned teachers and interested students at the levels of... high school or early undergraduate mathematics." As with Pólya's earlier books, the text follows the logic of the subject matter, placing the burden of inquiry on the student; the student learns by working sequences of "interesting and worthwhile problems." And as with Pólya's other books, the student who manages to work all the problems has done a fair amount of honest-to-goodness mathematics. *Mathematical Discovery* offers a wealth of problems from straightedge and compass geometric constructions, to algebra, recursion, and applications of superposition. Students who work those problems will have learned some important techniques in a way that gets to the structure of the underlying mathematics and that cuts through the standard topical presentations. The books also offer theoretical descriptions of the problem-solving process, supplemented with exercises so that students "live" the mathematics rather than just reading about it. In short, *Mathematical Discovery* is a classic. It's a required text in my undergraduate problem-solving classes and in my classes for high school mathematics teachers. It represents the capstone of Pólya's career, returning to the themes he explored throughout that career: the structure of mathematics and the nature of mathematical discovery. The combined volumes on those topics—the two problem books co-authored with Szegő, *How to Solve It*, the two volumes of *Mathematics and Plausible Reasoning*, and the two volumes of *Mathematical Discovery*—comprise a collection of extraordinary scope and power.

Heuristics and problem solving since 1945

This section traces developments in mathematical problem solving since 1945. Since problem solving and education lie in the public arena, there are two dimensions along which to consider those developments: the impact of Pólya's work and ideas in the real world and the evolution of research on problem solving, considered as a field of scientific inquiry.

Indicators of Pólya's influence on the real world are plentiful. For example, the National Council of Teachers of Mathematics recommended in its (1980) *Agenda for Action* that "problem solving be the focus of school mathematics in the 1980's." To help move things along the 1980 NCTM yearbook [4] was devoted to *Problem Solving in School Mathematics*; subsequent yearbooks and other NCTM publications followed up on the same theme. To get the problem-solving bandwagon rolling, the 1980 Yearbook featured the four-stage problem-solving model from *How to Solve It* on its inside covers. Moreover, virtually every article in the Yearbook is based on Pólya's ideas. In short, "problem solving" in mathematics education means problem solving à la Pólya. And the problem-solving movement is hardly limited to mathematics education; if anything, it's nearly universal. In the past five years Pólya's work on problem solving has been cited in the *American Political Science Review*, *Annual Review of Psychology*, *Artificial Intelligence*, *Computers and Chemistry*, *Computers and Education*, *Discourse Processes*, *Educational Leadership*, *Higher Education*, *Human Learning*, and many more journals. Pólya's real-world impact is clear.

In contrast, the scientific status of heuristic strategies—Do people really use them? Do they work? Can they be taught?—has been problematic.* On the one hand, such strategies have enormous face validity. My own reaction to them was typical. A practicing mathematician, I tripped across Pólya's writings on mathematical thinking about a dozen years ago. I read through them with increasing wonder. Page after page, Pólya described the problem-solving techniques I had learned to use—but had never been taught!—as I did mathematics. In Pólya's writings I recognized both the ways that I did mathematics and the ways that I thought about it. That recognition was shared by most mathematicians, and by most mathematics educators. Pólya's discussions just felt right.

They weren't, however. Or to put it another way, there was ample empirical evidence to suggest that something was either wrong or missing. Excited by my readings in the early 1970s, I sought out some problem-solving experts, mathematics faculty who coached students for the Putnam exam or for various Olympiads. Their verdict was unanimous and unequivocal: Pólya was of no use for budding young problem-solvers. Students don't learn to solve problems by reading Pólya's books, they said. In their experience, students learned to solve problems by (starting with raw talent and) solving lots of problems. The mathematics education literature was no more encouraging. Largely because Pólya's ideas seemed so right, the math-ed literature was chock full of studies designed to teach problem solving *via* heuristics. Unfortunately, the results—whether in first grade, algebra, calculus, or number theory, to name a few—were all depressingly the same, and served to confirm the statements of the Putnam and Olympiad trainers. Study after study produced "promising" results, where teacher and students alike were happy with the instruction (a typical phenomenon when teachers have a vested interest in a new program), but where there was at best marginal evidence (if any!) of improved problem-solving

*Parts of this section are adapted from Schoenfeld [13], where more details may be found. The truly masochistic reader can find full documentation for the ideas sketched here in my *Mathematical Problem Solving* [12].

performance. Despite all the enthusiasm for the approach, there was no clear evidence that the students had actually learned more as a result of their heuristic instruction, or that they had learned any general problem-solving skills that transferred to novel situations. Finally, Pólya's ideas were challenged on scientific grounds by researchers in artificial intelligence (AI). In their classic book *Human Problem Solving* [7], Newell and Simon describe the genesis of a computer program called General Problem Solver (GPS). GPS was developed to solve problems in symbolic logic, chess, and "cryptarithmetic" (a puzzle domain similar to cryptograms, but with letters standing for numbers instead of letters). GPS played a decent game of chess, solved cryptarithmetic problems fairly well, and managed to prove almost all of the first 50 theorems in Russell and Whitehead's *Principia Mathematica*—all in all, rather convincing evidence that its problem-solving strategies were pretty solid. Now for researchers in AI, the proof is in the empirical pudding: your theory is right if (and, the hard-nosed AI guys would say, only if) you have a running program that instantiates your theory and solves the problems you claimed it would. By those standards, heuristics à la Pólya fell short. One leading AI researcher put it bluntly: "We tried to write problem-solving programs using Pólya's heuristics, and they failed; we tried other methods, and they succeeded. Thus we suspect the strategies he describes are epiphenomenal rather than real—and even if they are real, they're far less important than the ones we use in our programs."

In sum: As of the mid-to-late 1970s, there was empirical reason, both in AI and mathematics education, to doubt the solidity of the heuristical foundations established by Pólya. In the past decade, things have begun to look brighter for heuristics—partly because some of the "general problem-solving techniques" promoted by early AI researchers didn't pan out as hoped, and partly because AI itself (more generally, cognitive science) provided some of the means by which Pólya's ideas could be refined and implemented. What follows is an outline of some developments since the mid-1970's. For details, see Schoenfeld ([**12**], [**13**]) and Silver [**15**].

At a certain level, Pólya's descriptions of problem-solving strategies were right. If you already knew how to use the strategies, you recognized them in his writings. That's why mathematicians resonated with Pólya's work. But at a finer grain size (that is, subjected to a more detailed analysis), Pólya's problem-solving descriptions didn't contain enough detail for people unfamiliar with the strategies to be able to implement them. Pólya's characterizations were *labels* under which families of related strategies were subsumed. One quick example will give the flavor of the analysis. The basic idea is that when you look closely at any single heuristic "strategy," it explodes into a dozen or more related, but fundamentally different, problem-solving techniques. Consider a typical strategy, "examining special cases":

To understand an unfamiliar problem better, you may wish to exemplify the problem by considering various special cases. This may suggest the direction of, or perhaps the plausibility of, a solution.

Now consider the solution to the following three problems.

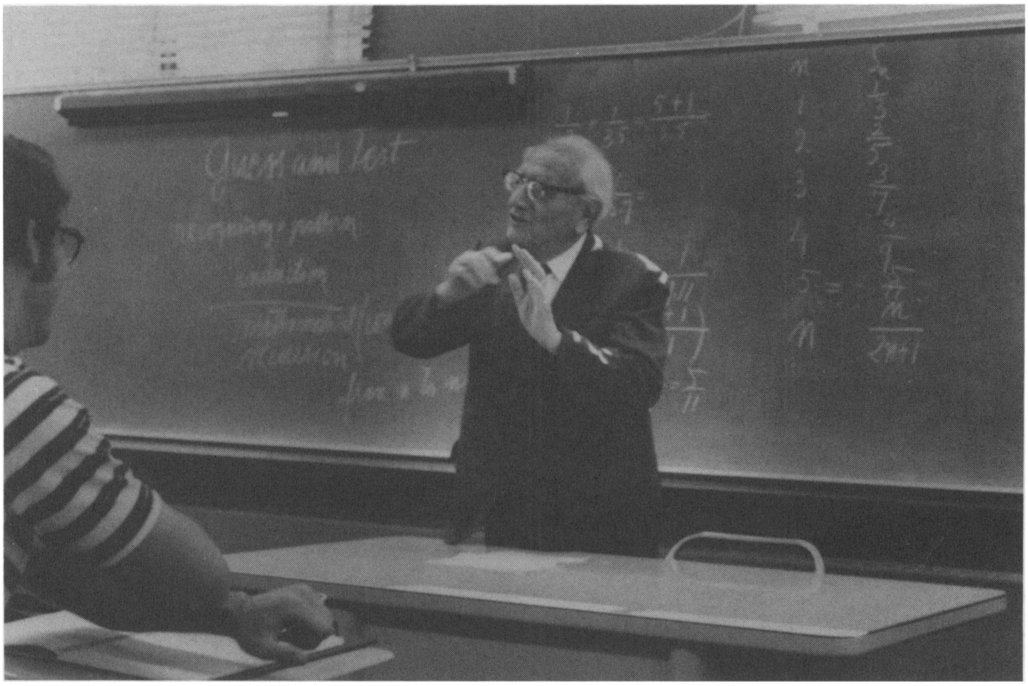
Problem 1. Determine a formula in closed form for the series

$$\sum_{k=1}^{n} k/(k+1)!$$

Problem 2. Let $P(x)$ and $Q(x)$ be polynomials whose coefficients are the same but in "backwards order":

$$P(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n, \quad \text{and}$$
$$Q(x) = a_n + a_{n-1} x + a_{n-2} x^2 + \cdots + a_0 x^n.$$

What is the relationship between the roots of $P(x)$ and $Q(x)$? Prove your answer.



Pólya lecturing at Santa Clara, c. 1978

Problem 3. Let the real numbers a_0 and a_1 be given. Define the sequence $\{a_n\}$ by

$$a_n = 1/2(a_{n-2} + a_{n-1}) \qquad \text{for each } n \geq 2.$$

Does the sequence $\{a_n\}$ converge? If so, to what value?

I'll leave the details of the solutions to you. However, the following observations are important. For problem 1, the special cases that help are examining what happens when the integer parameter, n, takes on the values $1, 2, 3\ldots$ in sequence; this suggests a general pattern that can be confirmed by induction. Yet if you try to use special cases in the same way on the second problem, you may get into trouble: Looking at values $n = 1, 2, 3, \ldots$ can lead to a wild goose chase. It turns out that the right special cases of $P(x)$ and $Q(x)$ to look at for problem 2 are easily factorable polynomials. If, for example, you consider

$$P(x) = (2x+1)(x+4)(3x-2),$$

you will discover that its "reverse," Q, is easily factorable. The roots of P and Q are easy to compare, and the result (which is best proved another way) is obvious. And again, the special cases that simplify the third problem are different in nature. If you choose the values $a_0 = 0$ and $a_1 = 1$, you can see what happens for that particular sequence. The pattern in that case suggests what happens in general, and (especially if you draw the right picture!) leads to a solution of the original problem.

Each of these problems typifies a large class of problems, and exemplifies a different special cases strategy. We have:

Strategy 1. When dealing with problems in which an integer parameter n plays a prominent role, it may be of use to examine values of $n = 1, 2, 3\ldots$ in sequence, in search of a pattern.

Strategy 2. When dealing with problems that concern the roots of polynomials, it may be of use to look at easily factorable polynomials.

Strategy 3. When dealing with problems that concern sequences or series that are constructed recursively, it may be of use to try initial values of 0 and 1—if such choices don't destroy the generality of the processes under investigation.

Needless to say, these three strategies hardly exhaust "special cases." At this level of analysis—the level of analysis necessary for implementing the strategies—one could find a dozen more. This is the case for almost all of Pólya's strategies. In consequence, the two dozen or so "powerful strategies" in *How to Solve It* are, in actuality, a collection of two or three hundred less "powerful," but actually usable, strategies. These strategies *can* be taught—but the fact that there are so many of them causes a new problem. With three hundred techniques potentially at your disposal, you have to know which ones to use, and at what times. "Knowing" the right method won't do you much good, if you don't get around to using it. A brief anecdote illustrates the dilemma.

A number of years ago, I deliberately put the problem

$$\int \frac{x\,dx}{x^2-9}$$

as the first problem on a test of techniques of integration, to give my students a boost as they began the exam. With the substitution $u = x^2 - 9$, which I expected the students to use, you can knock the problem off in just a few seconds. Half of the students did this, and got off to a good start. But a fourth of them used the correct but quite time-consuming procedure of partial fractions—and because they spent so much time on the problem, they did poorly on the exam. Half of the rest used the trig substitution $x = 3\sin\theta$—also correct, but so time-consuming that they wound up very far behind and bombed the exam. It is interesting that the students who used the harder techniques showed they knew "more," or at least, more difficult mathematics than the ones who used the easy technique. But they also showed that "it's not just what you know; it's how and when you use it." It's nice when what you do is right, but it's much better when it's also *appropriate*.

The last sentence is too optimistic in tone. The fact is that most of what we do is wrong. That is, most of our guesses—when we're working *real* problems—turn out not to be right. That's natural; the problems wouldn't be problems, but exercises, if that weren't the case. But the hallmark of good problem solvers is that they don't get lost forever in pursuing their wrong guesses. They manage to reject the ones that turn out not to be fruitful, and (eventually!) to hone in on directions that seem to help. Research now indicates that a large part of what comprises competent problem-solving behavior consists of the ability to monitor and assess what one does while working problems, and to make the most of the problem-solving resources at one's disposal. It also indicates that students are pretty poor at this, partly because issues of "resource allocation during thinking" are almost never discussed. But there is evidence that when students get coaching in problem solving that includes attention to such things—when they are encouraged to think about issues like "What are you doing? Why are you doing it? How will it help you solve the problem?"—their problem-solving performance can improve dramatically. (The jargon term for the "resource allocation" aspect of thinking discussed in this paragraph is *metacognition*. For a general introduction to the topic and for more details, the reader might want to look at the chapter entitled "What's all the fuss about metacognition?" in Schoenfeld [**14**].)

To sum up in a sentence, empirical evidence gained over the past decade indicates that Pólya's intuitions may have been right—at least when they're fleshed out with the details made accessible by the tools of cognitive science. So things look better for Pólya on the math-ed front. And on the AI front, things look better as well. In the 1970s researchers in AI were betting on powerful, domain-independent problem-solving strategies such as "means-ends analysis"; domain-specific strategies such as mathematical heuristics were not held in high regard. But the domain-independent strategies have not lived up to their promise, and current cognitive research focuses on the elaboration of problem-solving strategies tied to bodies of subject matter—e.g., on problem-solving strategies for particular domains of mathematics. We may be close to the point where we can build computer programs employing Pólya-type heuristic strategies. If the research gets to that point—and it may well, in the next decade or two—then Pólya's intuitions about problem solving will have served as part of the foundation for a true "science of thought."

An expanded version of the article, entitled "A brief and biased history of problem solving," appears as a chapter in: F. Curcio, (ed.), Teaching and Learning: A Focus on Problem Solving, National Council of Teachers of Mathematics, Reston, VA, 1987.

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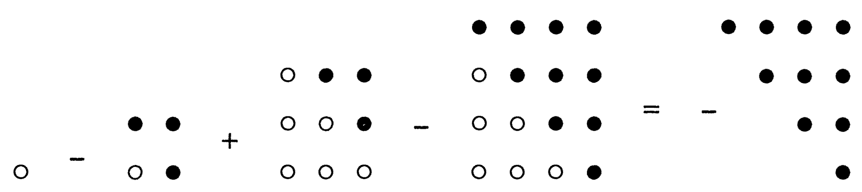
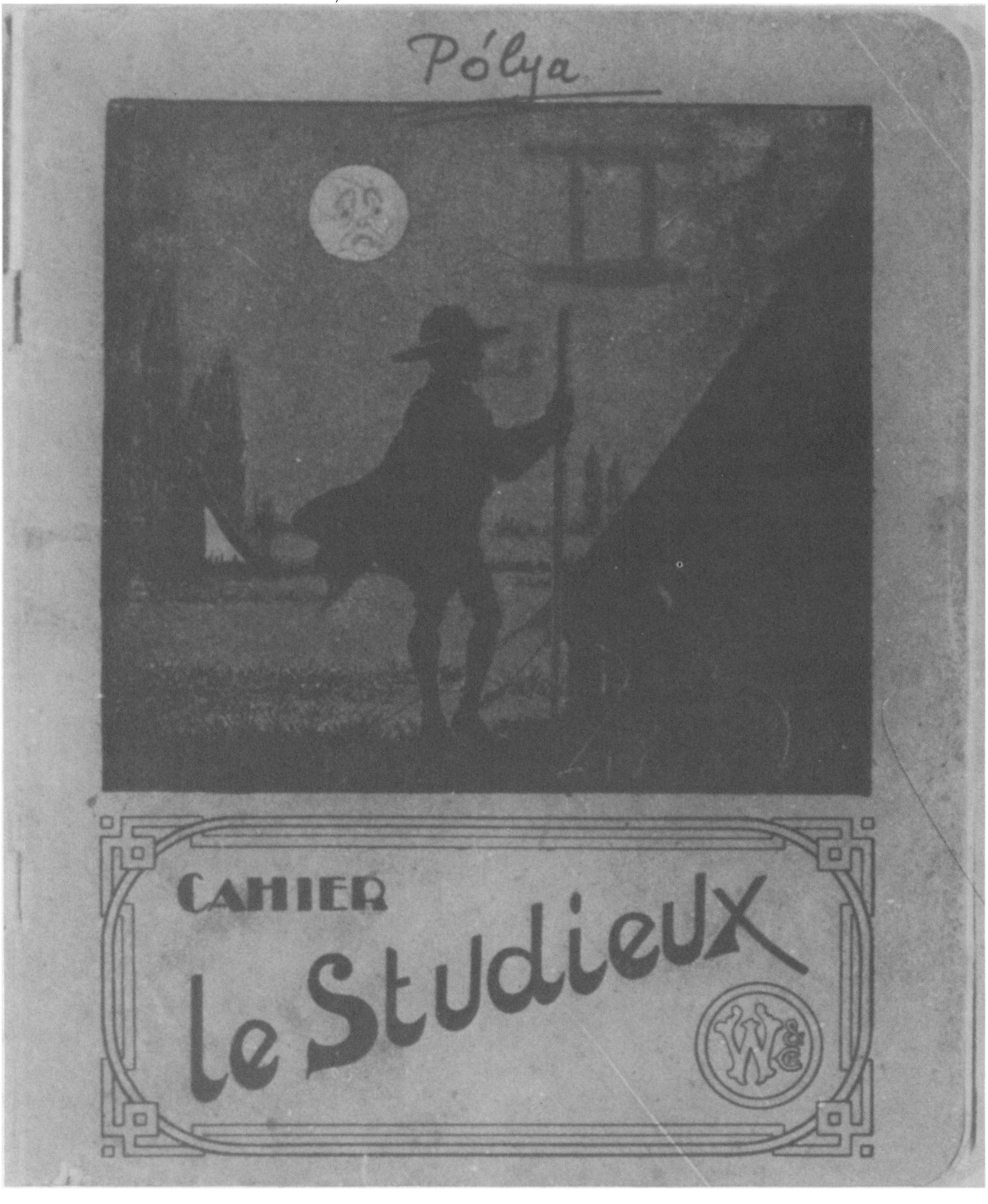


$$\sum_{k=1}^{n} (-1)^{k+1}k^2 = (-1)^{n+1}T_n = (-1)^{n+1}\frac{n(n+1)}{2}.$$

—Dave Logothetti
Santa Clara University


Pólya
CAHIER
le Studieux

The Pólya-Escher Connection

Doris Schattschneider
Moravian College
Bethlehem, PA 18018

Pólya: The single word is written in Escher's hand on the cover of the small school-notebook. The second of 17 numbered workbooks filled with sketches and investigations on regular plane division by Escher, this one was singled out by the artist for future reference.

Mathematicians have admired and used the work of the Dutch graphic artist M. C. Escher (1898–1972) ever since his work became widely known to the mathematical community. Many encountered his work for the first time when an exhibition was held at the Stedelijk Museum in Amsterdam in conjunction with the 1954 International Congress of Mathematicians. There is no doubt that some of his work after that time was directly the result of fruitful interchanges with mathematicians. (H. S. M. Coxeter, Roger Penrose, and Caroline MacGillavry are among those whose ideas were appropriated by Escher with spectacular results.)

Long before these contacts, however, Escher had benefitted from interaction with the work of other mathematicians—most notably, Pólya. In 1922, Escher visited the Alhambra in Granada, Spain, for the first time. He wrote

> The fitting together of congruent figures whose shapes evoke in the observer an association with an object or a living creature intrigued me increasingly after that first Spanish visit in 1922 ... I periodically returned to the mental gymnastics of my puzzles. In about 1924 I first printed a piece of fabric with a wood block of a single animal motif which is repeated according to a particular system, always bearing in mind the principle that there may not be any "empty spaces." ... I exhibited this piece of printed fabric with my other work, but it was not successful. [**1**, p. 55]

Unknown to Escher at the time, in this same year George Pólya published his paper *Über die Analogie des Kristallsymmetrie in der Ebene* [**8**] in which he classified the 17 plane symmetry groups. Pólya was not the first to publish this classification (E. S. Fedorov had published it in 1891) but, true to his own teaching maxim "make a picture," Pólya provided a full-page illustration that gave an example of a tiling for each of the 17 groups (Figure 1). He was quite proud of this illustration, using some favorite classical tilings and creating others himself.

Escher's crude first attempts at tessellation and his disappointment with them did not quell his desire to continue and to succeed.

> In the beginning I puzzled quite instinctively, driven by an irresistible pleasure in repeating the same forms, without gaps, on a piece of paper. These first drawings were tremendously time-devouring because I had never heard of crystallography; so I did not know that my game was based on rules which have been scientifically investigated. [**7**, p. vii]

In 1936 Escher visited the Alhambra for the second time, and also the mosque La Mezquita, in Córdoba. This time he made careful colored drawings of the many geometric tessellations he found there. These studies were a constant source of ideas—later many served as geometric skeletons which Escher caused to metamorphose into tilings of butterflies, fish, lizards, and birds. Also about this time, Escher's brother, B. G. Escher, a geologist at the University of Leiden, brought to Escher's attention several articles, including Pólya's, published in *Zeitschrift für*

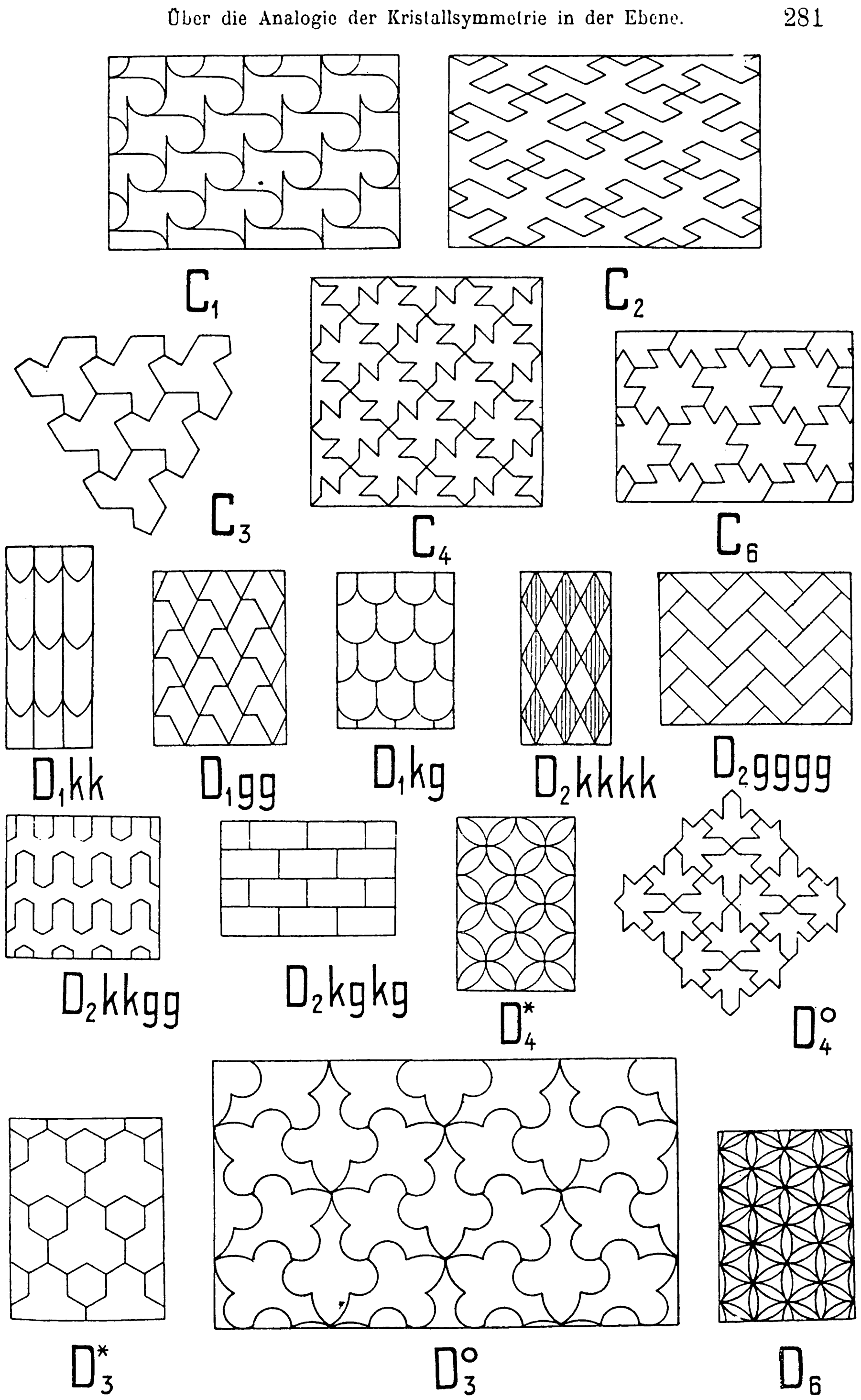

Über die Analogie der Kristallsymmetrie in der Ebene. 281
C_1
C_2
C_3
C_4
C_6
D_1kk
D_1gg
D_1kg
D_2kkkk
D_2gggg
D_2kkgg
D_2kgkg
D_4^*
D_4°
D_3^*
D_3°
D_6

FIGURE 1

Kristallographie. Escher copied Pólya's article in full—the text, dutifully recorded, was probably of little use to Escher. Pólya spoke of groups, and was concerned with classification. Escher was interested in gleaning the geometric essentials for creating periodic tilings. He records in his notebook each of Pólya's 17 tilings, just two to a page, and it is clear that these were not merely copied, but studied.

Following the Pólya paper, Escher's notebook records the bibliographic information on a paper by F. Haag [5]. Escher copied only one sentence of the text, Haag's definition of a regular plane tiling, from that paper. The tilings by polygons shown in that paper, as well as variants on them devised by Escher, fill several more notebook pages.

The medieval Islamic mathematical craftsmen, and the mathematicians Pólya and Haag, all spoke to Escher in the same language—the language of geometric figures. Escher's unusual receptivity to mathematical ideas and uncanny ability to discern the essential geometric relationships from this visual information allowed him to create the periodic designs which were his passion. Escher admitted that he "seriously tried to understand" the crystallographic literature he studied, "But they were mostly too difficult for my untrained mind and on the other hand they took no account of shade contrasts which for me are indispensable." [7, p. vii] This last comment is especially interesting, for in 1937 (when Escher recorded Pólya's paper) crystallographers had not yet considered "antisymmetry," or more generally, "color symmetry," the study of groups of isometries acting on colored tilings and the action of the isometries on the colors of the tiles.

Perhaps influenced by his observations at the Alhambra, Escher insisted that his tilings be map-colored. But more than that, from the very beginning of his studies (which culminated in his own formulation of a theory of color symmetry) he wove together the action of coloring the periodic designs with the symmetries that created the designs. As a result, his designs were inevitably "perfectly colored." (In algebraic terminology, a tiling with colored tiles is perfectly colored if the symmetry group which acts on the uncolored tiling induces a permutation of the set of colors of the tiles.)

Escher studied Pólya's tilings to understand their geometric structure, that is, how each tile was related to each of its neighbors. But his study went beyond this — he also considered how these tilings could be colored with a minimum number of colors in a way that was compatible with the symmetries of the tiling. Pólya had "colored" just one of his tilings, the diamond tiling labeled D_2kkkk. Escher added color to his copies of four other tilings from Pólya's paper, and each one of these is a perfect coloring. His coloring of Pólya's fleur-de-lis design (D_3^0) with three colors is shown in FIGURE 2.

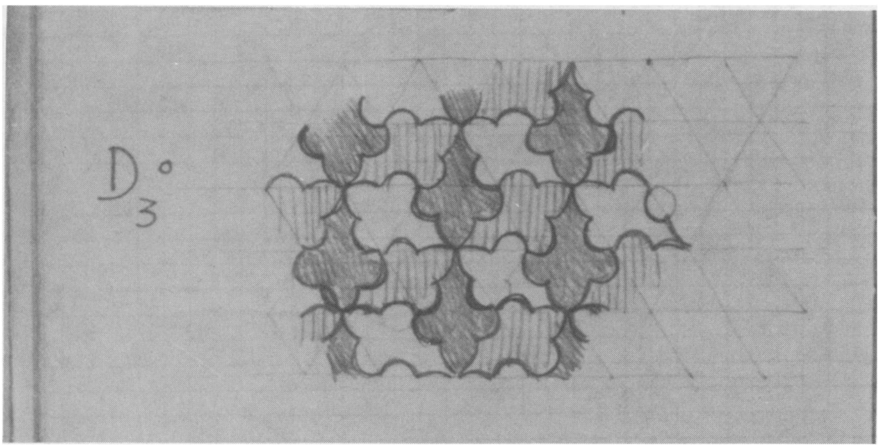


FIGURE 2

In November, 1937, Escher received from his brother the addresses of mathematicians to whom he wished to write about his work. The address "Prof. Dr. G. Pólya, Dunantstrasse 4, Zürich" was the first on the list (P. Niggli and A. Speiser were also named). Sometime after that, Escher wrote to Pólya to tell of his work, and a correspondence began, which included Escher's sending drawings to Pólya. Unfortunately, none of that correspondence seems to have survived. Pólya left the ETH in Zürich in 1940, and abandoned materials in his office, including the Escher correspondence. In 1976, at the Gemeentemuseum in The Hague, I saw the notebook containing Escher's hand-copied version of Pólya's paper and took a photo of it which I sent to Pólya. He was delighted since he had no material evidence of the Pólya-Escher connection. In 1977 Pólya told me of his active correspondence with Escher.

In looking at Pólya's illustrative tilings, it is not hard to imagine several of these turning into Escher tilings of birds, reptiles, and fish. One of these tilings was definitely the inspiration for an Escher design of eagles. Pólya's tiling labeled D_1gg (pg in modern crystallographic notation; see [9]) was copied by Escher, then changed to a tiling by blocks of two of Pólya's tiles fused into a single tile. Then gradually the blocks were transformed into eagles. FIGURE 3 shows an early stage

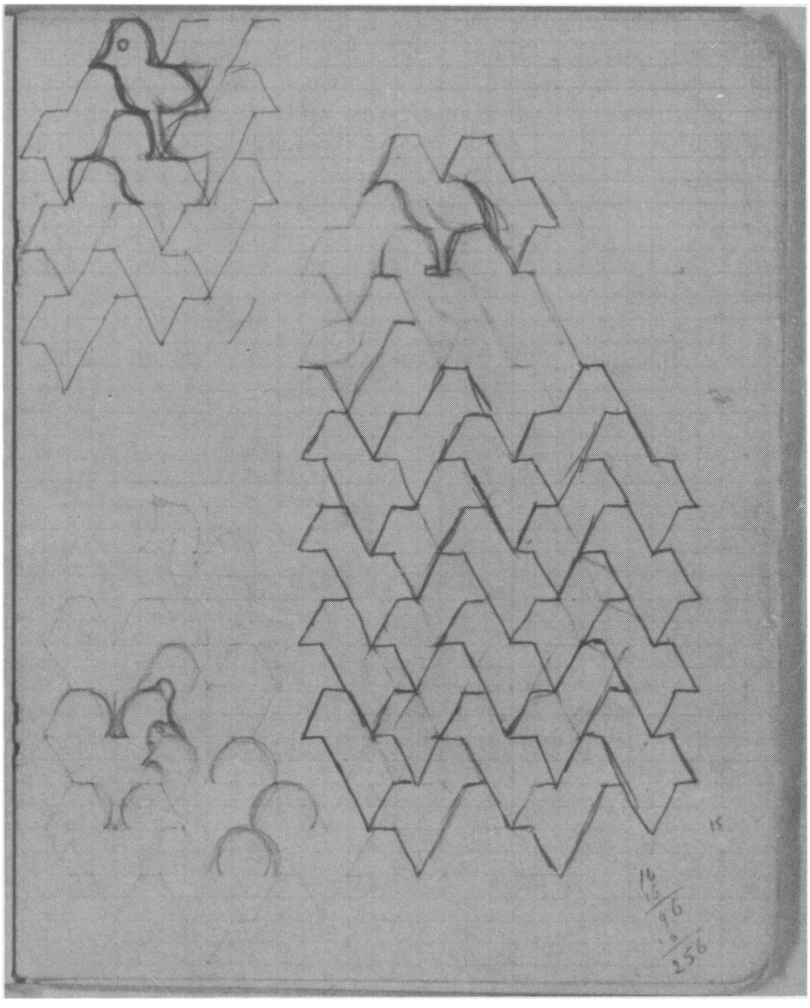


FIGURE 3

in this metamorphosis. The final design, number 17 of Escher's finished colored periodic designs (dated winter '38), is perfectly colored in red, white, and blue (FIGURE 4). The notation at the bottom of the drawing reads "overgangs systeem V^C-IV^B"; Escher's classification according to his own system.

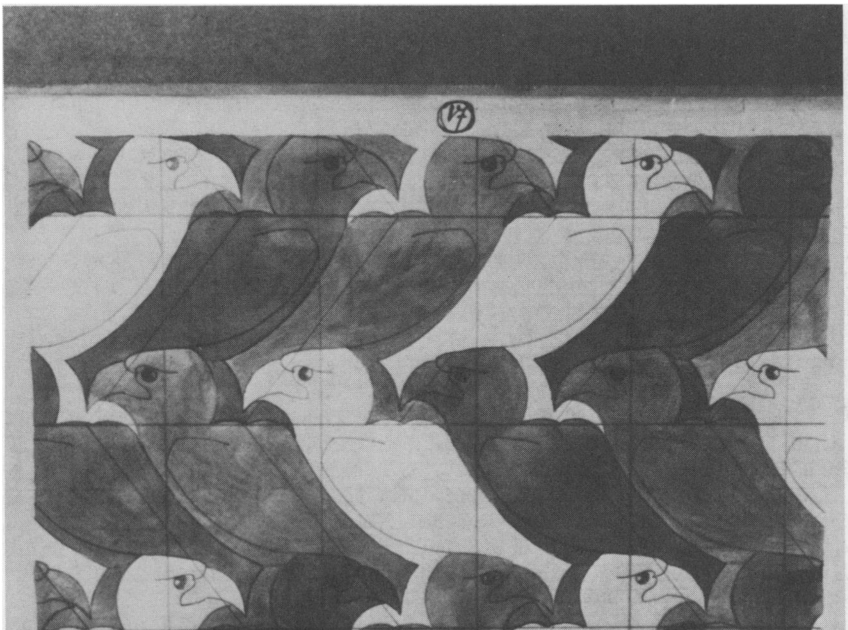


FIGURE 4

Pólya, in publishing pictures (17 of them) in 1924, gave visual instruction to Escher, providing him with the geometric keys to explore and create intricate and surprising plane tilings by fanciful creatures. Pólya's stimulation of Escher has come full cycle as Escher's periodic designs provide stimulation and challenge today to students and mathematicians alike.

I wish to express my appreciation to the staff of the Haags Gemeentemuseum for their cooperation and assistance in my research, and to Hans Cornet for his translations. I thank W. F. Veldhuysen, Cordon Art, for permission to reproduce Escher's work here, and Michael Sachs for providing the photo in FIGURE 4.

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[John von Neumann] is the only student of mine I was ever intimidated by. He was so quick. There was a seminar for advanced students in Zürich that I was teaching and von Neumann was in the class. I came to a certain theorem, and I said it is not proved and it may be difficult. Von Neumann didn't say anything but after five minutes he raised his hand. When I called on him he went to the blackboard and proceeded to write down the proof. After that I was afraid of von Neumann.

G. Pólya, *The Pólya Picture Album/Encounters of a Mathematician*, Birkhäuser, 1987, p. 154

George Pólya's Influence on Mathematics Education

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A century from now, George Pólya will be known through his books, papers, and a handful of videotapes and films of his teaching. No one then alive will have been one of his students or colleagues. What will his influence be on future generations of mathematics teachers? How will mathematics be taught differently because Pólya considered how people learn and do mathematics, and then he spoke and wrote and taught?

Currently, Pólya's most obvious influence on mathematics education is through his ideas on how problems can be solved. "Problem solving" has become the watchword among trendy mathematics educators, and George Pólya is their guru. The modest list of questions and suggestions for solving problems that he put forward in *How to Solve It* [**2**] has been transformed into a set of definitive "problem-solving strategies" that both describe human "information processing" for researchers and prescribe for teachers what students should be taught to do. The four *phases* of problem solving into which he grouped his observations—understanding the problem, devising a plan, carrying out the plan, and looking back—have metamorphosed into four *stages* through which problem solvers pass. The "Pólya model" has become a ritual learned in teacher education courses and passed on to classes of baffled children, who wonder, for example, why anyone would want to look back at the banal problems with which they are beset.

The theme of the 1980 yearbook of the National Council of Teachers of Mathematics [**1**] was problem solving, and when the advisory panel for the yearbook met to examine the more than 80 proposals for articles, they found references to Pólya in almost every one. Pólya has become the Marx and Lenin of mathematical problem solving; a few words of obeisance need to be offered in his name before an author can get down to the topic at hand.

This spate of misreading and sanctification has, not surprisingly, led some iconoclasts to reject much of what is understood to be the message of Pólya's writings. We are told that Pólya's advice on heuristics is useless: Real mathematicians do not follow it, students cannot learn from it, researchers cannot get measurable results from it, and perhaps most appalling of all, computers cannot depend upon it in their attempts to simulate intelligent behavior. Pólya is criticized for not having put his "strategies" into operational form, for neglecting in his "theory" the importance of conceptual knowledge, for overlooking the uses of mathematics in "real" situations, for failing to provide a complete and foolproof system. In other words, he was unable to make problem solving a science.

The fact that Pólya did not attempt to make problem solving a science but rather considered it, like teaching, a practical art often gets lost amid the homage and the disparagement. Pólya drew upon his own experience as a teacher and as a problem solver in offering his advice. He knew better than anyone that it provided neither system nor theory. That is one reason he set up part of it as "A Short Dictionary of Heuristic" [**2**, pp. 37-232]. Dictionaries require the reader to employ them selectively, as they become intelligible and connected to what the reader is thinking about. They amplify the reader's thoughts; they do not dictate the form or arrangement of those thoughts. Pólya saw his advice as providing some guidance to the baffled problem solver—some encouragement to stay with a problem, try to figure out what it is about, try to develop a reasonable plan for attacking it, and try to learn something useful from your experience with it.

Pólya considered the solution of problems to be the backbone of mathematics instruction. But, as he pointed out:

There are problems and problems, and all sorts of differences between problems. Yet the difference which is the most important for the teacher is that between "routine" and "nonroutine" problems. The nonroutine problem demands some degree of creativity and originality from the student, the routine problem does not... I shall not explain what is a nonroutine mathematical problem: If you have never solved one, if you have never experienced the tension and triumph of discovery, and if, after some years of teaching, you have not yet observed such tension and triumph in one of your students, look for another job and stop teaching mathematics. [**3**, pp. 126–127]

Pólya's attention to the art of teaching was like his attention to the art of problem solving: He gave examples and suggestions, not theories or recipes. His influence on the teaching of mathematics has been subtle. Pólya held that imitation and practice are the principal means through which problem solving is learned. Books and films can provide examples of problems being solved and of problems to solve, but no words or pictures can replace the experience of being in a class where the teacher solves problems with you and then gives you an opportunity to practice what you have learned. Although mathematics teachers have been quick to adopt what they understand to be Pólya's approach to solving problems, and although they find many of his problems engrossing, few have been able to alter their instruction and recast the curriculum to reflect his challenging pedagogical ideas.

Perhaps Pólya's most profound influence will ultimately be a transformation not only in what it means to teach mathematics but also in our view of mathematics itself. All of Pólya's efforts to help teachers and students derived from the view that we understand mathematics best when we see it being born, by either following the steps of historical discoveries or engaging in discoveries ourselves. He wanted students to see the scaffolding of mathematics under construction, not simply the finished product. He saw mathematics in the making as an inductive science, yielding its secrets through clever guess, followed by careful test, followed by refined guess. Again and again, he used the power of the specific example to illuminate (and usually to help establish) the generalization. Philosophers of mathematics are now coming to accept Pólya's view of mathematics as practice: the idea that the experience of learning and doing mathematics defines our subject [**4**].

Subsequent generations will have the perspective needed to establish Pólya's place in the pantheon of mathematics education. We who knew him are too close, too confused, and too opinionated. But we have the better of it—we knew him.

References

[1] S. Krulik and R. E. Reys, eds., Problem Solving in School Mathematics, 1980 Yearbook, National Council of Teachers of Mathematics, Reston, VA, 1980.

[2] G. Pólya, How to Solve It, 2nd ed., Princeton, 1957.

[3] ______, On teaching problem solving, The Role of Axiomatics and Problem Solving in Mathematics, Conference Board of the Mathematical Sciences, Ginn, Boston, 1966, pp. 123–129.

[4] T. Tymoczko, ed., New Directions in the Philosophy of Mathematics: An Anthology, Birkhäuser, Boston, 1985.

[Hilbert] once had a student in mathematics who stopped coming to his lectures, and [he] was finally told that the young man had gone off to become a poet. Hilbert is reported to have remarked: "I never thought he had enough imagination to be a mathematician."

G. Pólya, *The Pólya Picture Album/Encounters of a Mathematician*, Birkhaüser, 1987, p.30

Constructing an Old Extreme Viewpoint

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In gratitude to George Pólya, who was the premier teacher in my life—from my days as his diligent and absorbent reader in graduate school until his death.

The old "picture on the wall" problem has been solved with varying insights over the years. It goes like this:

A picture hangs on a wall above the level of an observer's eye. Where should the observer stand in order to maximize the angle of observation of the picture?

This note records an easy *construction* of the solution "viewpoint" which yields the extremum sought. (For examples of treatments, see [**1**] and [**2**], where each treatment contains helpful elements not contained in the other.)

We first restrict our attention to the situation shown in FIGURE 1, where the picture extends from B to C on a wall which is perpendicular at F to the horizontal line λ which indicates the level of the observer's eye. We let b and c be the distances from F to B and F to C, respectively, and we seek the point P on λ at which the angle BPC is a maximum. Our easily constructed solution viewpoint is shown as point A in FIGURE 2. We first show how to locate A, and then show that it is indeed the solution point.

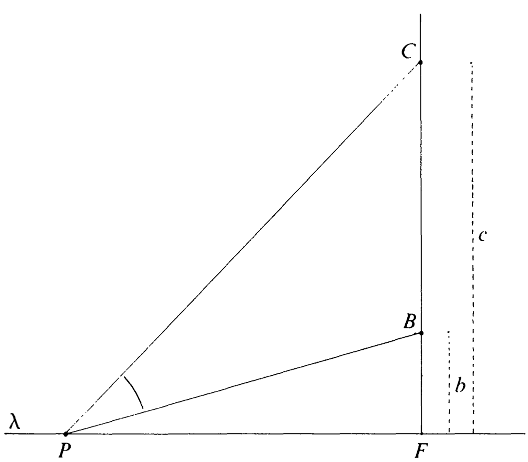

FIGURE 1

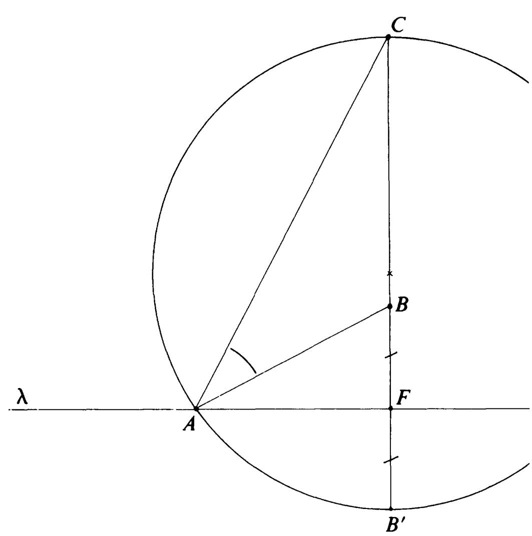

FIGURE 2

Constructing A in FIGURE 2. Extend the line CBF to B' as shown, where B and B' are both at a distance b from F. Then construct the circle which has CB' as its diameter. This circle intersects λ in the desired point A.

Showing that our point A in FIGURE 2 *is indeed the solution viewpoint.* First see FIGURE 3, which tersely recalls Pólya's use in [2] of the idea of "tangent level lines" to show that the solution point which we seek, call it T, is determined by that circle through B and C which is *tangent* to λ. (Of course, the three circles shown in FIGURE 3 are examples of the *contours*—the "level lines"—along each of which the angle of observation of the painting is a constant, this angle decreasing as the circles grow in diameter. The contour that helps us solve our problem is the one that is *tangent* to λ.) Then, following Niven in [1], we recall an old geometry theorem about tangents and secants which says that, in FIGURE 3, $FT \cdot FT = FB \cdot FC$; since our $FB = b$ and $FC = c$, we conclude that $FT = \sqrt{bc}$. That $\sqrt{bc}$ is indeed the length of the segment AF in FIGURE 2 follows easily from FIGURE 4a, where we recall how to construct $\sqrt{xy}$ given x and y. (To be clear: Given the lengths x and y, we construct the circle with diameter $x+y$, and then erect the perpendicular shown, which is easily seen to have the length $\sqrt{xy}$.) Voilà.

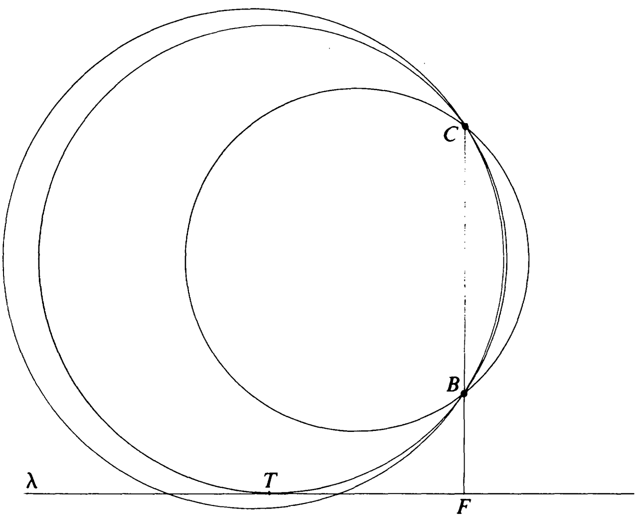

FIGURE 3

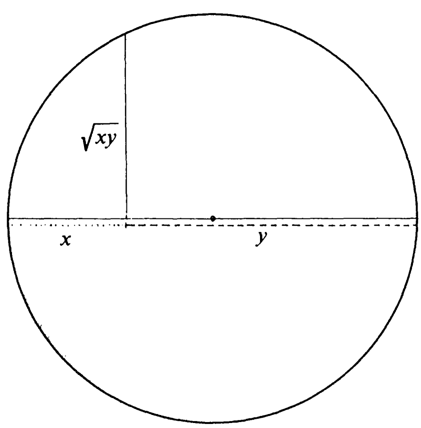

FIGURE 4a

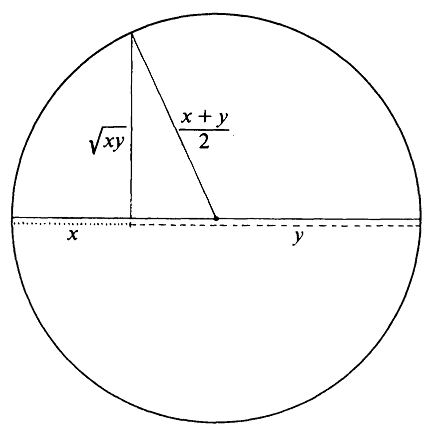

FIGURE 4b

[I record this personal aside: I never simply draw FIGURE 4a; I always go on to draw that radius in FIGURE 4b—because this provides an elegant and memorable pictorial demonstration of the geometric-arithmetic mean inequality which every generation should celebrate. It says that

$\sqrt{xy} \leqslant (x+y)/2$, with equality if and only if $x=y$. Pólya once told me that he had shown that picture "to $n+1$ teachers" in his time.]

If we now consider the case where the picture is tilted out from the wall, as shown in FIGURE 1′, we can still use tangent level lines and we can again use to advantage that old theorem about tangents and secants. In fact, the solution above is but a simple special case of what we now derive.

In parallel with what has gone before, our first job is to give a simple statement of the construction to be used, with a diagram in parallel with FIGURE 2. Thus, we consider FIGURE 2′. Here the line λ and the points B and C are given, and we easily get the point F' as the intersection of two lines; and the point B' by extending the line CBF' so that $F'B' = F'B$. Next we draw a semicircle on CB', and the perpendicular to CB' at F' which meets that semicircle at A'. We note that $F'A' = \sqrt{b'c'}$, for we've again used the familiar construction discussed in FIGURE 4a, and we finally mark off the length $F'A$ on the line λ equal in length to $F'A'$. Then A is the solution point we seek.

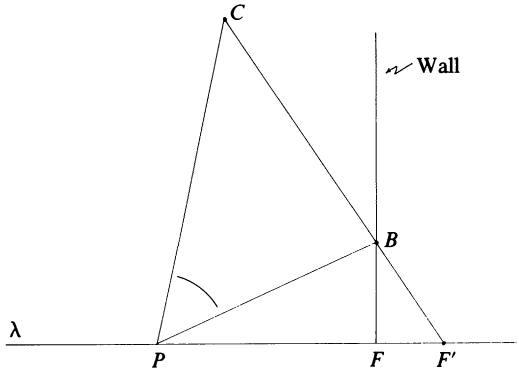

FIGURE 1′

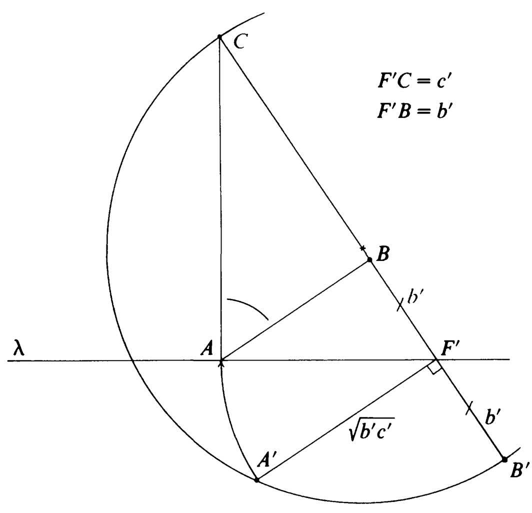

FIGURE 2′

Now, again in parallel with what we did earlier, we need a proof that the point A is indeed the solution point. We use FIGURE 3′. Here the required tangent level line—a circle through B and C —meets λ in T. Then $F'T = \sqrt{F'B \cdot F'C} = \sqrt{b'c'}$,—by that old theorem about tangents and secants—thus showing that the point T in FIGURE 3′ is the same as the point A constructed in FIGURE 2′. Encore voilà.

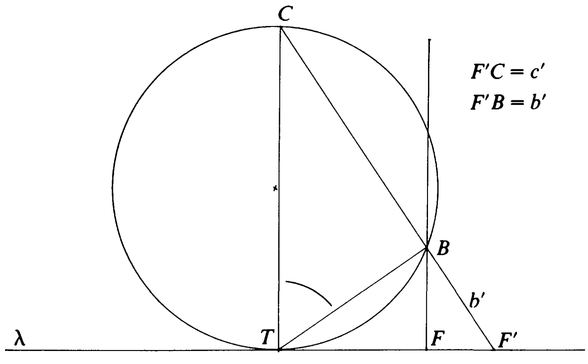

FIGURE 3′

Remark (a). It may be useful to consider a few more observations using FIGURE 5 and the notations introduced there. The tangency point T = A is the solution point we found. Using the theorem about tangents and secants, we have

$$(w + w')^2 = b'c'. \tag{I}$$

Solving (I) for w, a length *inside* the viewing room, and using easy trigonometry, we have

$$w = \sqrt{bc}/\cos\beta - b\tan\beta, \tag{II}$$

and, of course, if $\beta = 0$, then (I) and (II) both reduce to $w = \sqrt{bc}$ as expected. We note that the two lengths which make up w in (II) could easily be constructed, but the earlier procedures are much more satisfying.

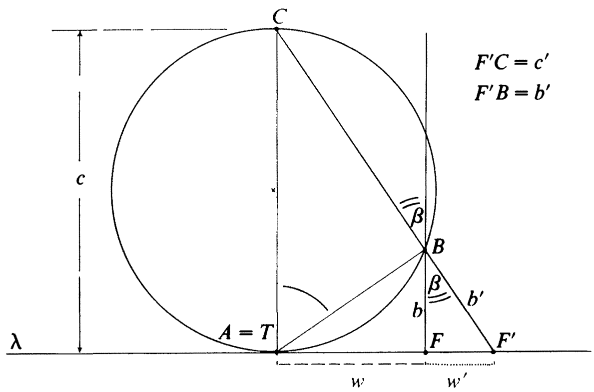

FIGURE 5

Remark (b). What happens if the picture is *not* "above the level of an observer's eye"? Answer: If the picture is entirely *below* λ, one simply reflects everything in that line and applies the argument above, of course. If, on the other hand, the line λ *intersects* the picture, then the viewing angle tends toward 0 as the observer moves away from the picture, and it tends to π as the observer moves toward the picture—and there is no maximum for the viewer to use short of crashing into the picture.

References

[1] Ivan Niven, Maxima and Minima Without Calculus, Dolciani Mathematical Expositions, no. 6, MAA, 1981, pp. 71–73.

[2] George Pólya, Induction and Analogy in Mathematics, Princeton, 1954, pp. 122 ff.

[Some] are obsessed by the idea of generalization. Everything should be generalized and their ideal seems to be a mathematical theorem of perfect generality, of such perfect generality that no particular consequence of it can be derived.

G. Pólya, in *Mathematical People/Profiles and Interviews* (D. J. Albers and G. L. Alexanderson, editors), Birkhäuser, 1985, p. 248

Looking into Pascal's Triangle: Combinatorics, Arithmetic, and Geometry

In grateful and affectionate remembrance of George Pólya

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Introduction

Since combinatorics is playing an increasingly important role in mathematics today, it is natural that the subject should take on an enhanced significance at the undergraduate level. The methods of combinatorics are varied, some being particular to the subject (e.g., graph theory) while in other cases traditional arguments based on classical arithmetic and algebra are appropriate. Fundamental to combinatorics are the binomial coefficients: in studying them and their properties we frequently have a choice of method. As a general rule we have found that algebraic arguments are more elementary and automatic, while combinatorial arguments convey greater insight.

We display in this article some patterns observed in Pascal's triangle and describe both our experimental method for making observations and our methods of proof. We would like especially to draw attention to the role of symmetry in our proof of Theorems 2.1 and 2.2, since the symmetries are not those traditionally associated with Pascal's triangle. The power of symmetry in mathematical proof is the theme of [**4**].

Certainly we have not even begun to exhaust the possibilities with respect to either the methods used or the results obtained. Indeed, one of our purposes in writing this article is to encourage others to look for further, and even more interesting, results. Is there, for example, a Bigger Hockey Stick and Puck Theorem (see Section 1)? And will there *always* be an Even Bigger one?

We have referred to the *left-justified* version of Pascal's triangle. For those (like PH, unlike JP) to whom this term is unfamiliar, let us explain that it refers to the deformation of the triangle which renders the left leg vertical. The advantages, in certain cases, of such a deformation illustrate the sense in which we claim that it is often geometric insight which leads to the conjectures for whose proof algebra* or combinatorics, or both, are required. We cannot do better than quote from Jacob Bernoulli on Pascal's triangle [**2**]: "The essence of combinations is concealed in it, but those who are more intimately acquainted with Geometry know also that capital secrets of all mathematics are hidden in it."

No one has been more successful, or more subtle, in demonstrating the truth of Bernoulli's dictum than George Pólya, who also saw the great possibilities for developing good problem-solving strategies which are inherent in Pascal's triangle. We have often taken our inspiration from Pólya's great work *Mathematical Discovery* [**6**]; this particular article was inspired by Section 3.6 et seq. of Volume 1 of that text.

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*Our point of view is that algebra, at this level, is simply systematized arithmetic. Hence, in this article, the two terms are interchangeable. See [**5**] for a more detailed statement of this point of view.

1. Some elementary discoveries in Pascal's triangle

The binomial coefficients lie at the heart of combinatorial mathematics. If r objects are to be selected without replacement from n distinct objects, then the number of ways this may be done is given by

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}. \tag{1.1}$$

This formula provides a fundamental link between combinatorics and arithmetic. Properties of the binomial coefficients may be proved, or exploited, by combinatorial or arithmetical reasoning, and we may also invoke geometrical ideas in studying them.

A good example of an arithmetical fact proved combinatorially is the following.

THEOREM 1.1. *The product of r consecutive integers is divisible by r!.*

Proof. It is plainly sufficient to consider the product of positive integers. Let k be the least of the r consecutive positive integers. Then the product is $k(k+1)\cdots(k+r-1)$ and

$$\frac{k(k+1)\cdots(k+r-1)}{r!} = \binom{k+r-1}{r}.$$

However, the right-hand side is, from its combinatorial interpretation, clearly an integer.

As a second example of this phenomenon we have the following.

THEOREM 1.2. (i) n divides $\binom{2n-2}{n-1}$;

(ii) $(2n-1)$ divides $\binom{2n-1}{n}$;

(iii) $(4n-2)$ divides $\binom{2n}{n}$.

Proof. The quotients are alternative forms of the nth Catalan number c_n, defined as the number of ways of inserting parentheses into the sum of n numbers so that the expression makes sense (thus $c_1 = 1,\ c_2 = 1,\ c_3 = 2,\ c_4 = 5,\ c_5 = 14, \ldots$).

No easy arithmetical proof of these theorems seems available. Often one may choose between combinatorial and arithmetical proofs; in such cases the combinatorial proof usually provides greater insight. An example is the *Pascal identity*.

$$\binom{n+1}{r} = \binom{n}{r} + \binom{n}{r-1}. \tag{1.2}$$

Of course this identity can be proved directly from (1.1), but the following argument seems more satisfactory: We know that

$$\binom{n+1}{r} = \text{the number of choices of } r \text{ objects from } n+1 \text{ objects.}$$

We now separate the choices into two disjoint classes. First,

$$\binom{n}{r} = \text{the number of choices excluding the last object,}$$

since we then select r objects from the first n objects; second,

$$\binom{n}{r-1} = \text{the number of choices including the last object,}$$

since we then select $(r-1)$ objects from the first n objects. Thus (1.2) follows.

Another example (involving a result which we will use later) concerns the binomial coefficient $\binom{n+1}{2}$. Geometrically, we may look at a "representative" case by examining a 5×5 square array of dots partitioned into three disjoint sets as shown in FIGURE 1.

We see from the figure that

$$2(1+2+3+4)+5=5^2,$$

from which it follows that

$$1+2+3+4=\frac{5(5-1)}{2}.$$

And we recognize this to be $\binom{5}{2}$.

Since for this argument the number 5 behaves in exactly the same way as other natural numbers ($\geqslant 3$), we readily see that the corresponding partitioning of an $(n+1)\times(n+1)$ array of dots leads to the identity

$$1+2+3+\cdots+n=\frac{(n+1)n}{2},$$

and the expression on the right is just $\binom{n+1}{2}$. It is now easy to *see* why this binomial coefficient is sometimes referred to as the nth *triangular* number, denoted T_n. We can carry this geometrical point of view even further. By partitioning our representative 5×5 array into *two* triangular arrays of dots, as in FIGURE 2, we observe that $T_4+T_5=5^2$, from which it follows that $\binom{5}{2}+\binom{6}{2}=5^2$. Our figure also shows that $T_5=T_4+5$, so that $\binom{6}{2}=\binom{5}{2}+5$. The corresponding general conclusions are $T_{n-1}+T_n=n^2$, or

$$\binom{n}{2}+\binom{n+1}{2}=n^2; \tag{1.3}$$

and $T_n=T_{n-1}+n$, or

$$\binom{n+1}{2}=\binom{n}{2}+n. \tag{1.4}$$

Notice that (1.4) may also be seen as just a special case of the Pascal identity (1.2).

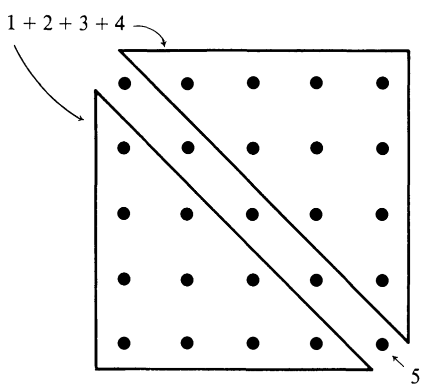


FIGURE 1

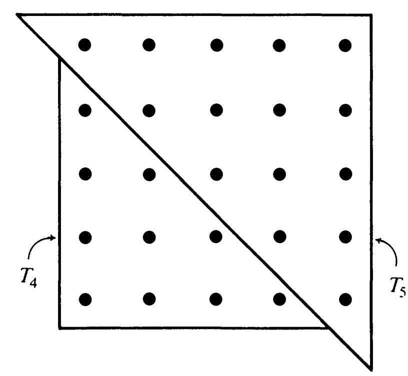


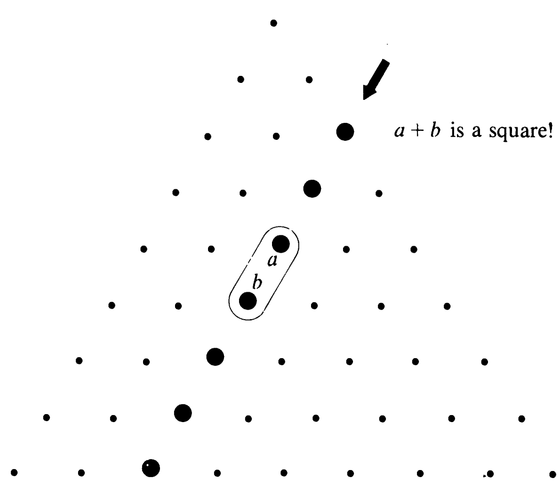
FIGURE 2

Pascal's triangle provides a laboratory for students to conduct mathematical experiments: looking for patterns, formulating from their observations some general statements, and then trying to prove them. In fact, it was a student in one of our classes who discovered the theorem below, which was new to us. Before giving that theorem, however, let us explain some of the ways we represent Pascal's triangle.

It is natural, in the early experimental stage, to look at the entries of Pascal's triangle as numbers.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1 (1.5)
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
1 9 36 84 126 126 84 36 9 1
1 10 45 120 210 252 210 120 45 10 1

However, once a pattern has been discovered, it is frequently useful to replace the numbers by dots. Thus, for example, the observation leading to the relationship (1.3) could be abbreviated by the following picture, where it is understood that the given equation holds for any pair of adjacent circled entries that appear along the particular diagonal indicated by the arrow.



(1.6)

FIGURE 3

Of course, since we are often obliged to give an arithmetical proof, it is also necessary to be able to think of Pascal's triangle expressed in terms of the actual binomial coefficients as follows.

$$
\begin{array}{ll}
\text{Row } 0 \rightarrow & \binom{0}{0} \\
\text{Row } 1 \rightarrow & \binom{1}{0}\ \binom{1}{1} \\
\text{Row } 2 \rightarrow & \binom{2}{0}\ \binom{2}{1}\ \binom{2}{2} \\
\text{Row } 3 \rightarrow & \binom{3}{0}\ \binom{3}{1}\ \binom{3}{2}\ \binom{3}{3} \\
\text{Row } 4 \rightarrow & \binom{4}{0}\ \binom{4}{1}\ \binom{4}{2}\ \binom{4}{3}\ \binom{4}{4} \\
\text{Row } 5 \rightarrow & \binom{5}{0}\ \binom{5}{1}\ \binom{5}{2}\ \binom{5}{3}\ \binom{5}{4}\ \binom{5}{5} \\
\text{Row } 6 \rightarrow & \binom{6}{0}\ \binom{6}{1}\ \binom{6}{2}\ \binom{6}{3}\ \binom{6}{4}\ \binom{6}{5}\ \binom{6}{6} \\
\text{Row } 7 \rightarrow & \binom{7}{0}\ \binom{7}{1}\ \binom{7}{2}\ \binom{7}{3}\ \binom{7}{4}\ \binom{7}{5}\ \binom{7}{6}\ \binom{7}{7} \\
\text{Row } 8 \rightarrow & \binom{8}{0}\ \binom{8}{1}\ \binom{8}{2}\ \binom{8}{3}\ \binom{8}{4}\ \binom{8}{5}\ \binom{8}{6}\ \binom{8}{7}\ \binom{8}{8} \\
\text{Row } 9 \rightarrow & \binom{9}{0}\ \binom{9}{1}\ \binom{9}{2}\ \binom{9}{3}\ \binom{9}{4}\ \binom{9}{5}\ \binom{9}{6}\ \binom{9}{7}\ \binom{9}{8}\ \binom{9}{9} \\
 & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot
\end{array}
\qquad (1.7)
$$

The element in the nth row on the rth diagonal is $\binom{n}{r}$.

When using Pascal's triangle as a mathematical laboratory, a natural course of events would be the following. First, a pattern is observed in the array (1.5), then it is expressed, for informal communication, as shown in (1.6) and, finally, stated as a conjectured theorem by replacing the picture involving dots by that involving the appropriate general binomial coefficients, displayed in (1.7). In most cases it is then simply a (sometimes tedious) matter to check the validity of the conjectured theorem by means of straightforward algebra using known facts like (1.1) or (1.2). It is our natural curiosity to understand *why* a relationship makes sense, coupled with a healthy desire to avoid tedium, that motivates us to look for combinatorial and geometrical interpretations of our results.

We now give some elementary examples. The following observation was first shown to us by Allison K. Fong, then a freshman at Santa Clara University. (The truth of the statement is preserved when the circled pattern is moved rigidly, backward or forward, in the direction indicated by the arrow.)

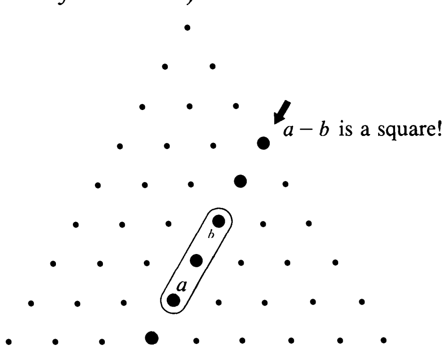


FIGURE 4

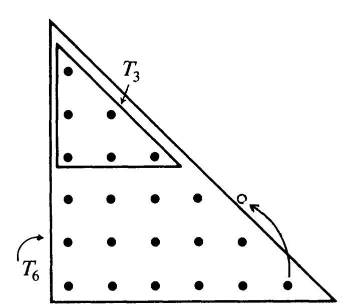


FIGURE 5

FONG'S THEOREM: *Indeed, referring to the binomial coefficients, and observing the value of the number whose square is $a-b$ for a few special cases, we formulate the conjecture*

$$\binom{n+2}{3}-\binom{n}{3}=n^2. \tag{1.8}$$

Proving this result by using (1.1) is somewhat tedious, but if we appeal to other known identities we quickly obtain

$$\begin{aligned}\binom{n+2}{3}-\binom{n}{3}&=\binom{n+1}{2}+\binom{n+1}{3}-\binom{n}{3}, \quad \text{by (1.2)}\\ &=\binom{n+1}{2}+\binom{n}{2}, \quad \text{by (1.2)}\\ &=n^2, \quad \text{by (1.3)}.\end{aligned}$$

An interesting but far from obvious route to this result, which combines the geometric and algebraic approaches, was suggested to us by the referee. Proceed, as before, by looking at the representative case $n=5$. FIGURE 5 shows that $T_6-T_3=3\times 5$, where T_n is the nth triangular number. In exactly the same way it can be shown, quite generally, that $T_{n+1}-T_{n-2}=3n$, $n\geqslant 3$. Thus

$$\frac{(n+2)(n+1)}{2}-\frac{(n-1)(n-2)}{2}=3n.$$

Multiplying by n and dividing by 3 yields

$$\frac{(n+2)(n+1)n}{1\cdot 2\cdot 3}-\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}=n^2,$$

which is (1.8).

We leave the proof of the following two theorems for the reader and simply present them in dot form, followed by the general statement in terms of binomial coefficients.

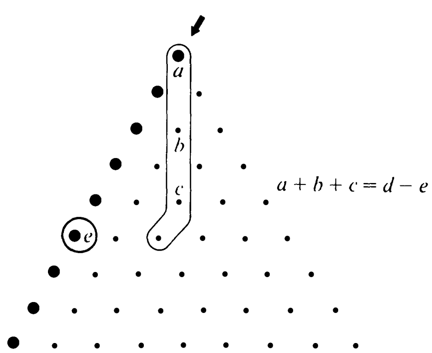


FIGURE 6

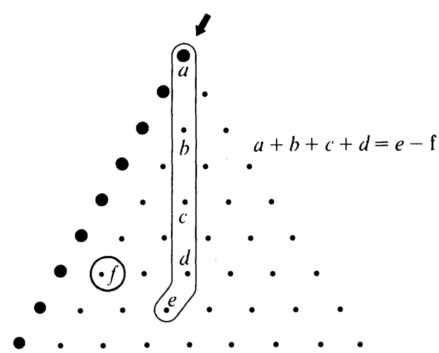


FIGURE 7

THEOREM 1.3 (*The Little Hockey Stick and Puck Theorem*)

$$\binom{n}{0}+\binom{n+2}{1}+\binom{n+4}{2}=\binom{n+5}{2}-\binom{n+5}{0}, \qquad n\geqslant 0.$$

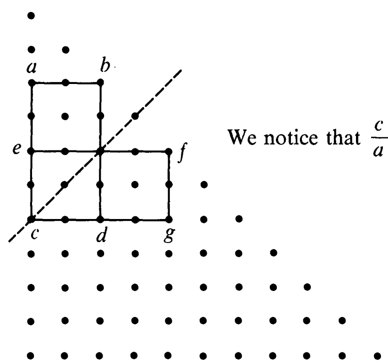
THEOREM 1.4 (*The Big Hockey Stick and Puck Theorem*).

$$\binom{n}{0}+\binom{n+2}{1}+\binom{n+4}{2}+\binom{n+6}{3}=\binom{n+7}{3}-\binom{n+6}{1}, \qquad n\geqslant 0.$$

We close this section, and motivate the deeper result of Section 2, by displaying, in dot form, some rectangular patterns in Pascal's triangle. We do not expect readers to observe these patterns immediately, any more than we did!

Notice that in these displays the entries of the triangle are *justified on the left*—thus they would take on the shape of certain special parallelograms in our original form of Pascal's triangle.

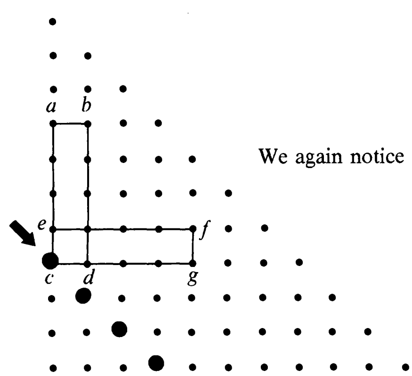
Our observation concerns rectangles of a fixed shape. In each of our pictures we display two such rectangles. In the first case they are both 2 units by 4 units, sharing the bottom left-hand corner. In the second case they are both 1 unit by 4 units, again sharing the bottom left-hand corner. Notice that, in each case, one rectangle is obtained from the other by reflection in the diagonal line of Pascal's triangle passing through their common corner. We may name two rectangles so related P and P'.



We notice that $\frac{c\times b}{a\times d}=\frac{c\times f}{e\times g}$

FIGURE 8

And, similarly,



We again notice that $\frac{c\times b}{a\times d}=\frac{c\times f}{e\times g}$

FIGURE 9

An even more remarkable observation is that the ratio $c\times b/a\times d$ is unaffected by sliding the rectangle in such a way that c moves along the diagonal (indicated by the arrow in FIGURE 9). This implies, of course, that the reflection property does *not* depend on the left-hand vertical side of the rectangle being along the edge of Pascal's triangle.

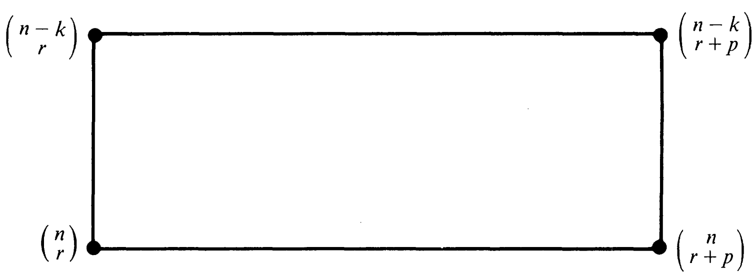
Before reading Section 2 (where the general phenomena are explained) the reader may wish to experiment with a few rectangles in the left-justified version of Pascal's triangle below. We give a few numerical examples in the early discussion in the next section.

n											
0	1										
1	1	1									
2	1	2	1								
3	1	3	3	1							
4	1	4	6	4	1						
5	1	5	10	10	5	1					
6	1	6	15	20	15	6	1				
7	1	7	21	35	35	21	7	1			
8	1	8	28	56	70	56	28	8	1		
9	1	9	36	84	126	126	84	36	9	1	
10	1	10	45	120	210	252	210	120	45	10	1
r	0	1	2	3	4	5	6	7	8	9	10

Pascal's triangle, left-justified

2. A deeper property of Pascal's triangle

We consider rectangles embedded in the left-justified version of Pascal's triangle with one pair of opposite sides horizontal and the other pair vertical. Then the vertices of the rectangle may be displayed as seen in FIGURE 10.



(2.1)

FIGURE 10

We obtain what we will call the *weight*, W, of the rectangle (2.1) as the result of 'cross-multiplication', precisely,

$$W = W(n, r, k, p) = \frac{\binom{n}{r}\binom{n-k}{r+p}}{\binom{n-k}{r}\binom{n}{r+p}}.$$

Our results are the following.*

THEOREM 2.1 (*The Reflection Property*). *The weight W is symmetric with respect to k, p; that is,*

$$W(n, r, k, p) = W(n, r, p, k).$$

THEOREM 2.2 (*The Sliding Property*). *For fixed k, p, the weight W depends only on $(n-r)$.*

Before proving these theorems, we illustrate their geometrical significance in Pascal's triangle. We remarked in discussing the examples at the end of Section 1 that, when Pascal's triangle is justified on the left, the weight of a rectangle is unaffected by interchanging the vertical and horizontal sides, fixing the lower left-hand vertex. For example, consider the rectangles in FIGURE 11:

*The vertical symmetry of Pascal's triangle in its original form implies that there are corresponding statements for rectangles in a *right*-justified Pascal's triangle.

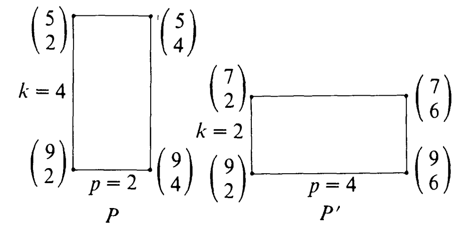


FIGURE 11

Then

$$W(P)=\frac{\binom{9}{2}\binom{5}{4}}{\binom{5}{2}\binom{9}{4}}=\frac{36\cdot 5}{10\cdot 126}=\frac{1}{7},$$

$$W(P')=\frac{\binom{9}{2}\binom{7}{6}}{\binom{7}{2}\binom{9}{6}}=\frac{36\cdot 7}{21\cdot 84}=\frac{1}{7}=W(P),$$

as claimed in Theorem 2.1.

THEOREM 2.2 asserts the other fact to which we drew attention about these rectangles, namely, that we may slide our rectangle up and down the diagonal $n-r=N$ (N fixed) without affecting its weight. For example, consider the rectangles in FIGURE 12:

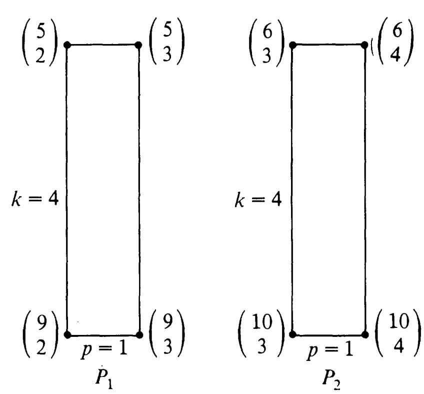


FIGURE 12

(Notice that P_2 is obtained from P_1 by sliding the vertex $\binom{9}{2}$ one place down the diagonal $n-r=7$.)

Then

$$W(P_1)=\frac{\binom{9}{2}\binom{5}{3}}{\binom{5}{2}\binom{9}{3}}=\frac{36\cdot 10}{10\cdot 84}=\frac{3}{7},$$

$$W(P_2)=\frac{\binom{10}{3}\binom{6}{4}}{\binom{6}{3}\binom{10}{4}}=\frac{120\cdot 15}{20\cdot 210}=\frac{3}{7}=W(P_1).$$

We now prove Theorems 2.1 and 2.2. We expand W as

$$W=\frac{n!}{r!(n-r)!}\cdot\frac{(n-k)!}{(r+p)!(n-r-k-p)!}\cdot\frac{r!(n-r-k)!}{(n-k)!}\cdot\frac{(r+p)!(n-r-p)!}{n!};$$

at this point we notice the possibility of some beautiful cancellation, producing the formula

$$W=\frac{(n-r-k)!(n-r-p)!}{(n-r)!(n-r-k-p)!}.$$

Then the expression on the right above is visibly symmetric in k and p and depends, for fixed k, p, only on $n-r$. This proves our two theorems.

We may ask whether the quantity $(n-r-k)!(n-r-p)!/(n-r)!(n-r-k-p)!$ has some combinatorial significance. We will answer this question affirmatively as follows. We write N for $n-r$, so that

$$W(N,k,p)=\frac{(N-k)!(N-p)!}{N!(N-k-p)!}.$$

We then observe that

$$W(N,k,p)=\frac{\binom{N-p}{k}}{\binom{N}{k}}=\frac{\binom{N-k}{p}}{\binom{N}{p}}. \tag{2.2}$$

The equality of the two fractions on the right of (2.2) reflects the equality

$$\binom{N}{k}\binom{N-k}{p}=\binom{N}{p}\binom{N-p}{k}, \tag{2.3}$$

which may be interpreted as follows. Suppose two people, A and B, are to choose from N objects; suppose further that A is to choose k objects, while B is to choose p objects. In how many ways may the choices be made? If we assume that A chooses first (an assumption which, of course, does not affect the number of possible choices) we find that the number of choices is given by the left-hand side of (2.3); while if we assume that B chooses first, the number of choices is given by the right-hand side of (2.3). This gives us a combinatorial proof of (2.3) which does not involve *any* arithmetic.

Remark. The special case of Theorem 2.1 given by $k=2$, $p=1$ may be familiar to some readers as the *Star of David* property of Pascal's triangle [3]. For, in that case, the identity $W(n,r,2,1)=W(n,r,1,2)$ effectively asserts that

$$\binom{n-2}{r+1}\binom{n-1}{r}\binom{n}{r+2}=\binom{n-2}{r}\binom{n}{r+1}\binom{n-1}{r+2}.$$

This is the Star of David centered at $\binom{n-1}{r+1}$,

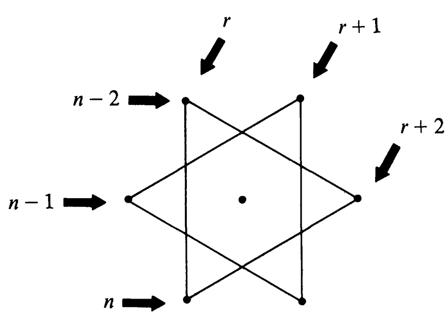


FIGURE 13

If we wish to center the star at $\binom{n}{r}$, the identity reads

$$\binom{n-1}{r}\binom{n}{r-1}\binom{n+1}{r+1}=\binom{n-1}{r-1}\binom{n+1}{r}\binom{n}{r+1}$$

which is equivalent to the assertion

$$W(n+1,r-1,2,1)=W(n+1,r-1,1,2).$$

Study of the proof of Theorems 2.1, 2.2 shows that their conclusions remain valid if we replace the binomial coefficients in Pascal's triangle by expressions $f(n)/f(r)f(n-r)$, for some fixed function f. There is, indeed, an important generalization in this direction about which it is worthwhile being explicit.

Let

$$f(n)=(1-q^n)(1-q^{n-1})\cdots(1-q)\quad \text{if } n\geqslant 1,\qquad f(0)=1,$$

and let us write

$$\begin{bmatrix} n \\ r \end{bmatrix}=\frac{f(n)}{f(r)f(n-r)},\qquad 0\leqslant r\leqslant n. \tag{2.4}$$

The expressions $\begin{bmatrix} n \\ r \end{bmatrix}$ are called *Gaussian polynomials*, or *q-analogues* of binomial coefficients, and play a key role in the theory of partitions (see [**1**] and [7]). It is by no means obvious that they *are* polynomials (in q), but this is proved by induction on n from the *Pascal q-identity* (compare this with (1.2))

$$\begin{bmatrix} n+1 \\ r \end{bmatrix}=q^r\begin{bmatrix} n \\ r \end{bmatrix}+\begin{bmatrix} n \\ r-1 \end{bmatrix}. \tag{2.5}$$

Anyone familiar with L'Hôpital's Rule will quickly see that, in fact,

$$\begin{bmatrix} n \\ r \end{bmatrix}\to\binom{n}{r}\quad \text{as } q\to 1. \tag{2.6}$$

Since Theorems 2.1, 2.2 remain true if we replace $n!$ by any $f(n)$, it follows, for example, that the Star of David property holds for Gaussian polynomials—a result which could be made to look rather deep!

Added in proof. We have recently noticed two further phenomena in the left-justified version of Pascal's triangle analogous to Theorems 2.1 and 2.2.

In the first we consider parallelograms with one pair of parallel sides *horizontal* and the other at a 45° angle to the horizontal. Their vertices are shown in FIGURE 14.

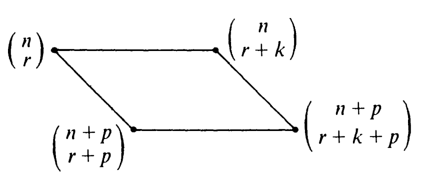


FIGURE 14

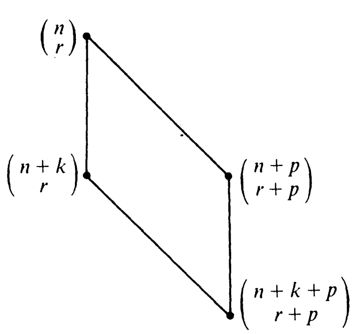


FIGURE 15

Then the weight of this parallelogram defined, as before, as

$$W=\binom{n+p}{r+p}\binom{n}{r+k}\Big/\binom{n}{r}\binom{n+p}{r+k+p},$$

turns out to be independent of n and symmetric in k and p.

In the second case we consider parallelograms with one pair of parallel sides *vertical* and the other at a 45° angle to the vertical. Their vertices are shown in FIGURE 15. This time the weight, defined as

$$W=\binom{n+k}{r}\binom{n+p}{r+p}\bigg/\binom{n}{r}\binom{n+k+p}{r+p},$$

turns out to be independent of r and symmetric in k and p.

These phenomena, perhaps surprisingly, are related to each other and to Theorems 2.1, 2.2 through the introduction of generalized binomial coefficients $\binom{n}{r}$ with n possibly negative. We plan to return to this theme.

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On Degree Achievement and Avoidance Games for Graphs

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To the loving memory of George Pólya on the occasion of the 100th anniversary of his birth.

Introduction

There is a family of constructive games concerning graphs which are all subsumed by the following procedure. Begin with p unlabeled isolated points in the plane which can be placed, without loss of generality, at the vertices of a regular polygon with p sides. There are two players, Alpha and Beta, who alternately add lines connecting pairs of points, with Alpha always moving first. Both players use the same color (pen or pencil if using paper; chalk if at a blackboard) so that it is not possible to tell who made which move. The first move by Alpha is to draw a line joining any two of the p points. As such a move always yields one line and $p-2$ isolated points, it cannot possibly have an effect on the outcome of the game, so it has been jokingly called **[3]** a *shrewd move*.

After this opening move, denoted by A_1, there are just two different possible replies which Beta can make. He can draw either a line having no points in common with the line of move A_1, or a line having just one point in common with the existing line. The game continues until either all $\binom{n}{2}$ lines have been drawn or until the "goal" has been reached, whichever comes first. Two of the goals for which winning strategies have been found are:

(a) diameter-2 graphs, studied by Buckley and Harary [**1**], and

(b) connected graphs, by Harary and Robinson [**4**].

Our present purpose is to develop strategies for playing "achievement and avoidance games" when the goal is a graph with a point of a given degree n.

We define an *achievement game* as one in which the first player (if any) to reach the specified goal wins. The corresponding *avoidance game* has the same goal and the same rules for making a "legal" move, but now the player who first reaches the goal loses; this has been called the *misère* version of the game.

We say that a player *wins* the degree n achievement or avoidance game if she can force a win regardless of the moves of her opponent. We treat only the avoidance games in detail here, since the player who wins the degree n avoidance game also wins the degree $n+1$ achievement game by moving to the point of degree n which his opponent is forced to create. Our purpose is to resolve the degree 3 avoidance game by determining the winner for all values of p.

Degrees 1 and 2

We follow the graph theory terminology and notation of the book [**2**]. In particular, K_p is the complete graph on p points, $\overline{K}_p$ is its totally disconnected complement (i.e., p isolated points), C_n is the cycle with n points, and P_n the path. Let A_k denote Alpha's kth move and B_k Beta's kth move. An asterisk (such as A_2^*) will denote a *forced move* (readily seen to be necessary to avoid losing forthwith).

The degree 1 achievement and avoidance games are completely trivial as Alpha, with shrewd move A_1, wins the achievement game and loses the avoidance version. The degree 2 achievement game is won by Beta on his first move by playing to one of the points where Alpha just moved.

The degree 2 avoidance game (and the degree achievement game) are only slightly less trivial. The players will alternate playing to points of degree 0 until forced to play to a point of degree 1. Thus Alpha wins when $p \equiv 2, 3 \pmod 4$ and Beta wins when $p \equiv 0, 1 \pmod 4$.

Degree 3 avoidance

The degree 3 avoidance game is more interesting. We will find it useful to develop some preliminary lemmas before proceeding to the general solution.

LEMMA 1. *When $p \geqslant 6$, Alpha can force the formation of one of the two graphs*: $C_4 \cup K_2 \cup \overline{K}_{p-6}$ *or* $C_5 \cup \overline{K}_{p-5}$.

Proof. For her first turn, Alpha makes the shrewd move of placing a line between any two points. Beta now has only two essentially different moves. He either plays a line incident with one of the two points Alpha just joined or he connects two other points. In either case, Alpha can form a path P_4 (FIGURES 1a and 1b). Now Beta cannot play to either point of degree 2 or he loses at once, so he has exactly three essentially different moves, each allowing Alpha to achieve one of the desired graphs (FIGURES 1c, 1d, and 1e).

LEMMA 2. *When $p \geqslant 5$, Beta can force the formation of one of the two graphs*: $C_3 \cup K_2 \cup \overline{K}_{p-5}$ *or* $C_4 \cup \overline{K}_{p-4}$.

Proof. After Alpha's first shrewd move, Beta forms a path P_3. Alpha then has exactly three essentially different moves, each of which allows Beta to form one of the two desired graphs (FIGURES 2a, 2b, and 2c).

We now treat the cases for $4 \leqslant p \leqslant 12$ individually. For $p \geqslant 13$ we will use the above observations to establish the general result. Note that the formation of a cycle C_m has the effect

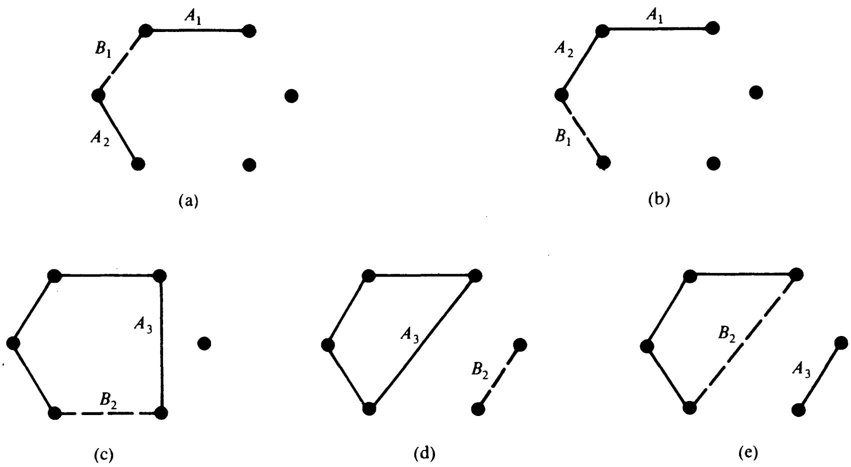

FIGURE 1. **Some opening moves in degree 3 avoidance: two graphs which Alpha can force.**

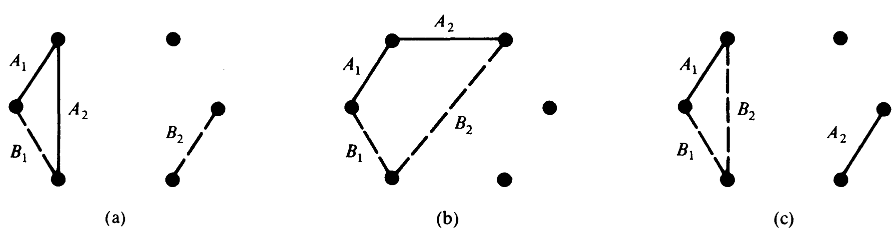

FIGURE 2. **Two graphs which Beta can force.**

of reducing the size of the board by m points since neither player can play to any point in the cycle without losing. As C_m also contains m lines, m moves are used to create the cycle. Thus, if m is odd, the formation of a C_m has the additional effect of reversing the roles of the first and second players, as in FIGURE 2(a) where Alpha formed a C_3 and Beta assumed the role of the first player on the remaining points.

Beta wins when $p = 4$

Proof. On his first move, Beta joins the two points which Alpha did not join. All following moves are forced.

Beta wins when $p = 5$

Proof. Apply Lemma 2. Whether Beta forms $C_3 \cup K_2$ or $C_4 \cup K_1$ on his second move, Alpha is forced to play to a point of degree 2. We note that Beta may also force a win if he plays to two new points on his first move and omit the details.

Alpha wins when $p = 6$

Proof. Apply Lemma 1. In either case, Beta is forced to play to a point of degree 2.

Alpha wins when $p = 7$

Proof. We obtain the result by demonstrating a winning strategy for Alpha. Beta has only two essentially different choices for his first move. If he plays to one of the points Alpha just joined, Alpha then forms a C_3, effectively removing three points from the board and assuming the role of the second player on the reduced board. Since the second player wins when $p = 4$, this would lead to a win for Alpha. Thus Beta is forced to play to two new points. This is really no better, as Alpha then plays to two of the remaining isolated points, leaving Beta two choices. The first, joining two points previously played to, allows Alpha to form a C_4, which may readily be seen to lead to a forced win (FIGURE 3a). In the second case, Beta plays to the last remaining isolated point. Alpha then forms a C_3, and the remaining moves are all forced (FIGURE 3b).

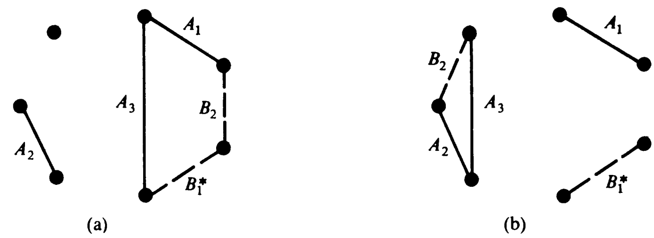

FIGURE 3. **Alpha wins when $p = 7$.**

Alpha wins when $p = 8$

Proof. We give a winning strategy for Alpha. On his first move, Beta is forced to play to two new points, for otherwise Alpha will form a cycle C_3, reducing the board to five points and assuming the role of the second player. For her second move, Alpha forms $P_3 \cup K_2 \cup \overline{K}_3$. Beta now has five essentially different choices. The first three allow the formation of a triangle, and Alpha easily wins the remaining game on five points (FIGURES 4a, 4b, and 4c). The remaining two moves allow Alpha to form $P_6 \cup \overline{K}_2$ (FIGURES 4d and 4e). In this case, Alpha wins on her next turn regardless of Beta's response by forming either $C_6 \cup K_2$ or $C_7 \cup K_1$. We note that Lemma 1 does not help Alpha here, since the formation of a C_4 leads to a loss.

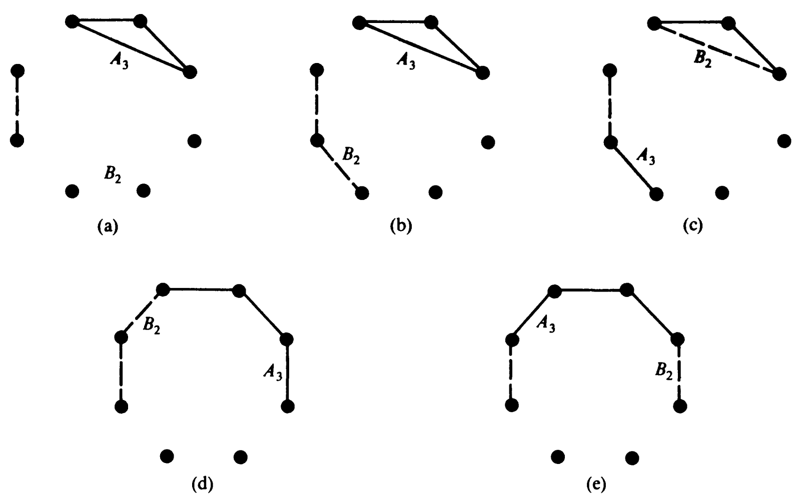

FIGURE 4. **Alpha wins when $p = 8$.**

Beta wins when $p = 9$

Proof. Apply Lemma 2. If Beta forces the formation of $C_3 \cup K_2 \cup \overline{K}_4$, then we have the same situation as when $p = 6$ with the roles of Alpha and Beta reversed. Since the first player wins when $p = 6$, this leads to a win for Beta. If Beta forces the formation of $C_4 \cup \overline{K}_5$, then he wins the reduced game on five points.

Alpha wins when $p = 10$

Proof. Apply Lemma 1. The argument is completely analogous to the argument for $p = 9$.

Beta wins when $p = 11$

Proof. Alpha's first three moves are forced (FIGURE 5a). As usual, A_1 is a shrewd move. Then A_2^* is forced, since otherwise Beta achieves a C_3, reducing the board to eight points and assuming the role of the first player. Finally, A_3^* is forced, lest Beta achieve a C_5, winning the remaining game on 6 points. On his third move, Beta forms $C_6 \cup \overline{K}_5$ and wins the remaining game on five points.

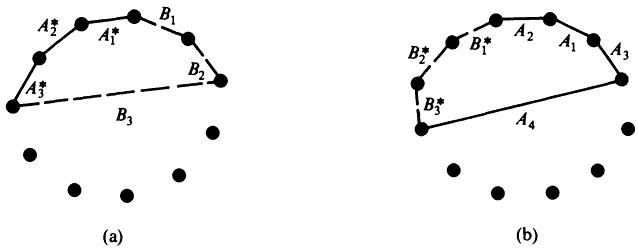

FIGURE 5. **Results for 11 and 12 points.**

Alpha wins when $p = 12$

Proof. This is very similar to the previous result. Beta's first three moves are forced (FIGURE 5b). First, B_1^* is necessary to prevent Alpha from forming a C_3 and winning the resulting game on 9 points. Then B_2^* is forced lest Alpha form a C_4. Finally, B_3^* is needed to prevent Alpha from forming a C_6. However, Alpha may now form a C_7, assuming the role of the second player on the remaining five points.

We now have enough consecutive cases of switching parity to use the first two lemmas to prove the general result.

THEOREM 1. *For all $p \geqslant 13$, Alpha wins when p is even and Beta wins when p is odd.*

Proof. Suppose the result is established for $p \leqslant m$, and consider the game played on m points with $m \geqslant 14$, even. Alpha can force the formation of one of the two graphs mentioned in Lemma 1. If Alpha forces $C_4 \cup K_2 \cup \overline{K}_{p-6}$ we have essentially four fewer points with Alpha still going first. Since Alpha wins for $p = m - 4$ (p even, $p \geqslant 10$), Alpha wins in this case. If Alpha forces $C_5 \cup \overline{K}_{p-5}$ we have essentially five fewer points with Beta going first. Since the second player is seen to win for odd $p = m - 5$, Alpha also wins in this case. Thus Alpha can always force a win when $p \geqslant 14$ is even.

Now consider the game played on m points with odd $m \geqslant 13$. The argument is completely analogous to that given above and uses Lemma 2. Here Beta can force a situation equivalent to $p = m - 3$ with the roles of Alpha and Beta reversed, or one equivalent to $p = m - 4$ with Alpha still going first. In either case, Beta wins.

Summary of results for degree 3 avoidance

Alpha wins when $p = 7$ or when p is even, $p \geqslant 6$.
Beta wins when $p = 4$ or 5, or when p is odd, $p \geqslant 9$.

Higher degrees

Increasing the degree from three to four results in a game which is still fun to play but is formidably more difficult to analyze. The above approach for degree 3 does not appear to extend to general degree n. The reason lies in the complexity of lifting Lemmas 1 and 2 to higher degrees. The maximal subgraphs with maximum degree 3 are much harder to characterize than those with maximum degree 2 which are unions of cycles together with at most one isolated point or with at most one pair of connected points.

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[2] F. Harary, Graph Theory, Addison-Wesley, Reading, 1969.
[3] F. Harary, Achievement and Avoidance Games, book in preparation.
[4] F. Harary and R. Robinson, Connect-it games, College Math. J., 15 (1984) 411–419.

Pólya's Geometric Picture of Complex Contour Integrals

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Complex contour integrals contain an element of mystery which has troubled me since my student days. Churchill and Brown [**1**] express the problem succinctly: "Definite integrals in calculus can be interpreted as areas, and they have other interpretations as well. Except in special cases, no corresponding helpful interpretation, geometric or physical, is available for integrals in the complex plane." In 1974, George Pólya suggested a simple solution, but his idea does not seem to be widely appreciated. Computer graphic techniques can be used to help students visualize and estimate complex integrals once Pólya's approach is adopted.

In classical potential theory it has long been the custom to associate to a real harmonic function $u(x, y)$ a "complex potential" $f(x+iy) = u(x, y) + iv(x, y)$, where $v(x, y)$ is a harmonic conjugate of $u(x, y)$. Then the real and imaginary parts of $\overline{f'(z)}$ are the components of the gradient field $\langle u_x(x, y), u_y(x, y)\rangle$ corresponding to the potential $u(x, y)$. Pólya's idea was simply this: to any complex function $f(x+iy) = u(x, y) + iv(x, y)$, associate the plane vector field $\overline{f(x+iy)} = \langle u(x, y), -v(x, y)\rangle$, rather than the derived field $\overline{f'(x+iy)}$. In [**2**] it is shown that complex integrals with integrand $f(z)$ have a simple geometric and physical interpretation in terms of the associated vector field $\overline{f(x+iy)}$. Our goal here is to spread this gospel, showing that the vector field picture can be used to estimate specific contour integrals, and leading to new insight into the theory of complex integration. An earlier paper [**3**] indicates the usefulness of the vector field picture of complex functions (as an alternative to the traditional view of a function as a mapping on the complex plane), in analyzing zeros and singular points of complex functions.

To emphasize the distinction between a complex function and its associated vector field, we henceforth write $\overline{\mathbf{W}}(z)$ or $\overline{\mathbf{W}}(x, y)$ to denote the Pólya vector field corresponding to a complex function $f(z)$. Thus if $f(x+iy) = u(x, y) + iv(x, y)$ is the decomposition of $f(z)$ into its real and imaginary parts, then $\overline{\mathbf{W}}(x, y) = \langle w_1(x, y), w_2(x, y)\rangle$ with $w_1 = u$, $w_2 = -v$.

The integral of f over an oriented curve γ can be expressed in terms of real integrals of the components of $\overline{\mathbf{W}}$ along γ:

$$\int_\gamma f(z)\,dz = \int_\gamma (u+iv)(dx+i\,dy) = \int_\gamma u\,dx - v\,dy + i\int_\gamma v\,dx + u\,dy$$

$$= \int_\gamma w_1\,dx + w_2\,dy + i\int_\gamma w_1\,dy - w_2\,dx = \int_\gamma \overline{\mathbf{W}}\cdot\mathbf{T}\,ds + i\int_\gamma \overline{\mathbf{W}}\cdot\mathbf{N}\,ds,$$

where $\mathbf{N}$ is the normal vector obtained by turning the unit tangent vector $\mathbf{T}$ *clockwise* through $\pi/2$. In words, the real part of $\int_\gamma f(z)\,dz$ is the integral of the tangential component of the Pólya vector field $\overline{\mathbf{W}}$ over γ (the flow along γ, if we picture $\overline{\mathbf{W}}$ as a velocity field); and the imaginary part of $\int_\gamma f(z)\,dz$ is the integral of the normal component of $\overline{\mathbf{W}}$ over γ (the *flux* across γ). An immediate payoff of this geometric interpretation is that it makes clear the fact that the value of a contour integral is independent of the parametrization, and changes sign if the orientation of the curve is reversed.

Just as one can estimate a real integral $\int_a^b f(x)\,dx$ by interpreting it as the signed area between the graph of f and the interval $[a, b]$ on the x-axis, a complex integral $\int_\gamma f(z)\,dz$ can be roughly approximated by visually estimating the flow and flux of the Pólya vector field $\overline{\mathbf{W}}$ along the path.

In FIGURE 1, for example, the vector field $\overline{\mathbf{W}}$ for the function $f(z) = 1/z$ is shown along the unit circle. The vector $\overline{\mathbf{W}}(z)$ is normal to the path at each point z, so the flow of $\overline{\mathbf{W}}$ along the contour is zero. The normal component of $\overline{\mathbf{W}}$ is apparently constant, namely 1, so the flux of $\overline{\mathbf{W}}$ across the path is simply this constant times the length of the path, viz., 2π. Thus our geometric analysis has shown that $\int_{|z|=1} \frac{1}{z}\,dz = 2\pi i$.

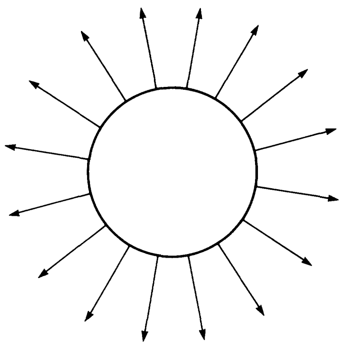

FIGURE 1. **Pólya vector field for $f(z) = 1/z$ on the unit circle.**

The example just considered is of course very special; in general one can only estimate the integrals of the tangential and normal components of $\overline{\mathbf{W}}$ over γ, from a plot of the vector field along this path. To emphasize how closely the procedure for estimating complex integrals parallels that for estimating real integrals, we ask the reader's indulgence as we briefly recall the latter.

To estimate $\int_a^b f(x)\,dx$ from a sketch of the graph of f over $[a, b]$, of course, one estimates the area between the graph and the x-axis, and subtracts the area below the axis from the area above. In more detail, we might mentally form a partition $a = x_0 < x_1 < \cdots < x_n = b$ such that $f(x)$ does not change sign on each subinterval; then estimate each of the integrals $\int_{x_k}^{x_{k+1}} f(x)\,dx$, and add the resulting signed numbers. To estimate the area between the graph and the x-axis over each subinterval $[x_k, x_{k+1}]$, we estimate the mean height $\overline{y_k}$ of the graph over this subinterval and use the product $\overline{y_k}(x_{k+1} - x_k)$ as our estimate of the area.

For example the mental process used in estimating $\int_0^5 f(x)\,dx$ for the function graphed in FIGURE 2 might go something like this: consider the partition $0 < 2 < 3 < 5$;

$$\int_0^2 f(x)\,dx \cong (1.1)(2-0), \qquad \int_2^3 f(x)\,dx \cong (-.1)(3-2), \qquad \int_3^5 f(x)\,dx \cong (.7)(5-3),$$

so $\int_0^5 f(x)\,dx \cong 2.2 - .1 + 1.4 = 3.5$. If an analytical evaluation produced a far different value for this integral, say -3, we would know an error had been made; the simplicity of the geometric estimate makes it very convincing.

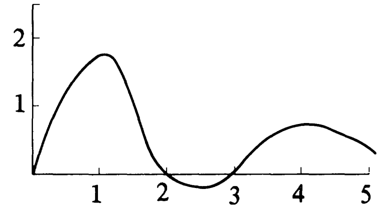

FIGURE 2

The vector field interpretation of complex integrals can be used in a similar way to provide a simple estimate based on geometric intuition, which can then be used as a check against analytical methods.

To estimate $\int_\gamma f(z)\,dz$, we must separately estimate its real part $\int_\gamma \overline{\mathbf{W}} \cdot \mathbf{T}\,ds$ and its imaginary part $\int_\gamma \overline{\mathbf{W}} \cdot \mathbf{N}\,ds$. To estimate $\int_\gamma \overline{\mathbf{W}} \cdot \mathbf{T}\,ds$, we first partition the curve into segments $\gamma_1, \gamma_2, \ldots, \gamma_n$ on which $\overline{\mathbf{W}} \cdot \mathbf{T}$ has constant sign (recall that $\overline{\mathbf{W}} \cdot \mathbf{T}$ is positive just if the angle between $\overline{\mathbf{W}}$ and $\mathbf{T}$ is acute). Then on each segment γ_k we visually estimate the mean tangential component τ_k of $\overline{\mathbf{W}}$, such that $\tau_k l_k \cong \int_{\gamma_k} \overline{\mathbf{W}} \cdot \mathbf{T}\,ds$, where l_k denotes the length of γ. In practice the plot of $\overline{\mathbf{W}}$ along γ is scaled, i.e., there is a factor SCALE such that a vector of apparent length 1 in the plot represents a vector in $\mathbb{C}$ with the same direction but of magnitude SCALE. Thus if τ_k denotes the *apparent* mean tangential component of $\overline{\mathbf{W}}$ along the curve segment γ_k, then $\int_\gamma \overline{\mathbf{W}} \cdot \mathbf{T}\,ds \cong$ SCALE $\sum_{k=1}^{n} \tau_k l_k$. The procedure for estimating $\int_\gamma \overline{\mathbf{W}} \cdot \mathbf{N}\,ds$ is similar.

EXAMPLE 1. If the plot of $\overline{\mathbf{W}}$ along γ were as indicated in FIGURE 3, to estimate $\int_\gamma f(z)\,dz$ we might reason as follows.

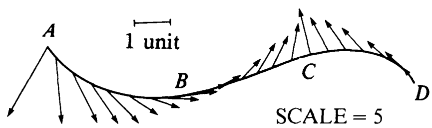

FIGURE 3

On segment AC the angle between $\overline{\mathbf{W}}$ and $\mathbf{T}$ is acute, so the tangential component of $\overline{\mathbf{W}}$ is positive on this segment. At A the (apparent) length of $\overline{\mathbf{W}}$ is about 2 units, and the tangential component (projection of $\overline{\mathbf{W}}$ onto the tangent line) is about 1.5 units. The vectors $\overline{\mathbf{W}}$ decrease in length as we move toward B, but they become more nearly parallel to $\mathbf{T}$, so the tangential component of $\overline{\mathbf{W}}$ decreases only to about .5 units. If we estimate the mean tangential component of $\overline{\mathbf{W}}$ to be 1 unit along this segment, then since the length of the arc AB is about 5 units, we estimate $\tau_1 l_1 \cong (1)(5) = 5$. The vectors $\overline{\mathbf{W}}$ increase in length from B to C, but the tangential component decreases from about .5 at B to 0 at C. Using an estimate $\tau_2 \cong .3$ for the mean tangential component on BC, and estimating the length of this segment to be 3 units, gives $\tau_2 l_2 \cong (.3)(3) = .9$. On segment CD the tangential component of $\overline{\mathbf{W}}$ starts at 0, becomes negative with a minimum of about $-.2$, and finally returns to 0 at D. We estimate $\tau_3 l_3 \cong (-.1)(3) = -.3$, and since the scale factor for the plot is SCALE = 5, our estimate of $\int_\gamma \overline{\mathbf{W}} \cdot \mathbf{T}\,ds$ would be

$5(5 + .9 - .3) = 28$. Similarly estimating the mean normal components of $\overline{\mathbf{W}}$ along AB, BC, and CD to be $\nu_1 \cong .5$, $\nu_2 \cong -.4$, and $\nu_3 \cong -.5$ gives SCALE $\sum \nu_k l_k \cong 5[(.5)(5) + (-.4)(3) + (-.5)(3)] = -1$. So if $\overline{\mathbf{W}}$ were the Pólya vector field of a complex function $f(z)$, our geometric estimates indicate that $\int_\gamma f(z)\,dz \cong 28 - i$. We cannot be certain of the sign of the imaginary part $\int_\gamma \overline{\mathbf{W}} \cdot \mathbf{N}\,ds$, since small errors in estimating the ν_k and l_k could affect the sign of the sum, but we can say with assurance that the flow along γ is positive (about 30), and the flux across γ is near 0. If an analytical calculation led to the result $\int_\gamma f(z)\,dz = 2\pi i$, we would have good reason to recheck the analysis.

In an introductory course on complex analysis the main "application" of complex integration is to evaluate certain real integrals using residue calculus. Typically one completes the real interval of integration to a closed contour in the complex plane, applies the residue theorem to evaluate an appropriate complex integral over this contour, and then tries to determine the contribution to the total produced by the integral along the real axis. By looking at a plot of the Pólya vector field along the contour, this last step can sometimes be clarified. (The residue at a simple pole also can be estimated geometrically, but showing how this may be done would take us too far from our main theme here.)

EXAMPLE 2. Evaluate

$$I_1 = \int_0^\infty \frac{1}{x^3 + 1}\,dx.$$

One first uses the residue theorem to evaluate

$$I = \int_\gamma \frac{1}{z^3 + 1}\,dz, \qquad \gamma = \gamma_1 + \gamma_R + \gamma_2,$$

where γ_1 follows the real axis from the origin to R, γ_R is the arc of the circle $|z| = R$ from R to $Re^{2\pi i/3}$, and γ_2 is the line segment from $Re^{2\pi i/3}$ back to the origin. $I = 2\pi i\,\mathrm{Res}(f, z_1)$, where $f(z) = 1/(z^3 + 1)$ and z_1 is the simple pole of f at $e^{\pi i/3}$. We calculate

$$\mathrm{Res}(f, z_1) = \lim_{z \to z_1} (z - z_1) f(z) = \frac{1}{\left[\frac{3}{2} + i\frac{\sqrt{3}}{2}\right](i\sqrt{3})}, \quad \text{so } I = \pi\left[\frac{1}{\sqrt{3}} - i\left(\frac{1}{3}\right)\right].$$

Because $|f(z)|$ decreases more rapidly than $\frac{1}{|z^2|}$ as $|z|$ increases, $\lim_{R\to\infty} \int_{\gamma_R} f(z)\,dz = 0$. So since the value of I is independent of R, $I = I_1 + I_2$, where $I_2 = \lim_{R\to\infty} \int_{\gamma_2} f(z)\,dz$.

Now the values of z^3 at $z = t$ on γ_1 and $z = te^{2\pi i/3}$ on γ_2 are identical, so the Pólya vectors $\overline{\mathbf{W}}(t)$ and $\overline{\mathbf{W}}(te^{2\pi i/3})$ are equal. However, $\overline{\mathbf{W}}(t)$ is directed along the path γ_1, whereas $\overline{\mathbf{W}}(te^{2\pi i/3})$ makes an angle of $\pi/3$ with the unit tangent vector $\mathbf{T}$ to γ_2. [See FIGURE 4.] So the tangential component of $\overline{\mathbf{W}}(te^{2\pi i/3})$ is

$$|\overline{\mathbf{W}}(t)|\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}|\overline{\mathbf{W}}(t)|,$$

and the normal component is

$$|\overline{\mathbf{W}}(t)|\cos\left(\frac{\pi}{2} + \frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}|\overline{\mathbf{W}}(t)|.$$

Thus

$$I_2 = \frac{1}{2}I_1 - i\frac{\sqrt{3}}{2}I_1, \quad \text{where } I_1 = \int_0^\infty \frac{1}{t^3 + 1}\,dt. \qquad (*)$$

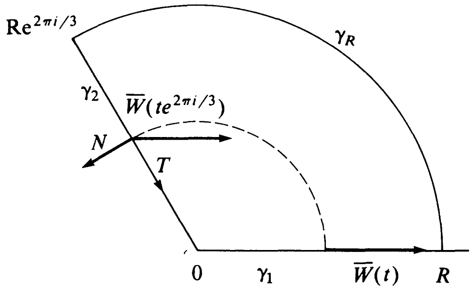

FIGURE 4

So

$$I = I_1 + I_2 = \left(\frac{3}{2} - i\frac{\sqrt{3}}{2}\right) I_1,$$

and comparing this with the value for I found above using the residue theorem, we conclude that $I_1 = \dfrac{2\pi}{3\sqrt{3}}$. Our discussion differs from the usual analytic evaluation of I in only one essential: rather than parametrizing γ_1 and deriving equation (*) by analytic means, our geometric argument uses the decomposition of $\overline{\mathbf{W}}(te^{2\pi i/3})$ into its tangential and normal components to derive this relationship between I_2 and I_1.

Besides clarifying the analysis of specific integrals, and throwing new light on familiar properties of complex functions, the vector field approach to complex integrals can lead to new theoretical results. The inequality $\left|\int_\gamma f(z)\,dz\right| \leqslant \int_\gamma |f(z)|\,ds$ is fundamental in estimating complex integrals, but does not seem to have a generally accepted name. It is sometimes called the *triangle inequality* for complex integrals, because it may be viewed as a consequence of the fact that the straight line segment is the shortest distance between two points in the complex plane. I have been unable to find any discussion in the literature of the conditions under which equality holds in this triangle inequality. The reason seems to be that the appropriate condition is not conveniently expressible in terms of the mapping properties of complex functions. But from the vector field point of view the condition is beautifully simple. Note that, because the value of a contour integral is unaffected when the path of integration is changed by a continuous deformation (keeping the endpoints fixed) in the region of analyticity of the integrand, the conditions for equality in the triangle inequality will involve both the contour γ and the integrand $f(z)$.

THEOREM. *Let $f(z)$ be a continuous complex function on a domain containing the piecewise differentiable arc γ. Then equality holds in the triangle inequality*:

$$\left|\int_\gamma f(z)\,dz\right| = \int_\gamma |f(z)|\,ds$$

exactly when the Pólya vector field $\overline{\mathbf{W}}$ *makes a constant angle with the tangent vector field* $\mathbf{T}$ *along* γ.

Our proof is based on a simple lemma about the modulus of a vector sum, and its continuous analogue for vector integrals.

LEMMA 1. *If* $\mathbf{W} = \sum_{k=1}^{n} \mathbf{V}_k$, *then* $|\mathbf{W}| = \sum_{k=1}^{n} |\mathbf{V}_k| \cos\theta_k$, *where* θ_k *is the angle between* $\mathbf{V}_k$ *and* $\mathbf{W}$. [*In words, the sum of the components of the summands along the sum* $\mathbf{W}$ *gives the modulus of the sum*, $|\mathbf{W}|$.]

Proof.

$$\sum_{k=1}^{n} |\mathbf{V}_k| \cos\theta_k = \sum_{k=1}^{n} \frac{\mathbf{V}_k \cdot \mathbf{W}}{|\mathbf{W}|} = \frac{1}{|\mathbf{W}|} \sum_{k=1}^{n} \mathbf{V}_k \cdot \mathbf{W} = \frac{1}{|\mathbf{W}|} \mathbf{W} \cdot \mathbf{W} = |\mathbf{W}|.$$

LEMMA 2. If $\mathbf{V}(t)$, $a \leqslant t \leqslant b$, is any continuous vector function, and $\mathbf{W} = \int_a^b \mathbf{V}(t)\,dt$, then $|\mathbf{W}| = \int_a^b |\mathbf{V}(t)| \cos\theta(t)\,dt$, where $\theta(t)$ is the angle between $\mathbf{V}(t)$ and $\mathbf{W}$.

Proof. Let $R_n = \sum_{k=1}^{n} \mathbf{V}(t_k)\Delta t$ denote a Riemann sum approximation to $\mathbf{W}$ relative to the partition of $[a, b]$ into n equal subintervals of length $\Delta t = (b-a)/n$. By Lemma 1, $|R_n| = \sum_{k=1}^{n} |\mathbf{V}(t_k)| \cos\theta_k \Delta t$, where θ_k is the angle between $\mathbf{V}(t_k)$ and R_n. As $n \to \infty$, $R_n \to \mathbf{W}$, so given any $\varepsilon > 0$, by taking n sufficiently large we can make $|\cos\theta_k - \cos\theta(t_k)| < \dfrac{\varepsilon}{M(b-a)}$ for all k, where $\theta(t_k)$ is the angle between $\mathbf{V}(t_k)$ and $\mathbf{W}$, and $M = \max\limits_{t\in[a,b]} |\mathbf{V}(t)|$. Then

$$\left|\sum_{k=1}^{n} |\mathbf{V}(t_k)|(\cos\theta_k - \cos\theta(t_k))\Delta t\right| \leqslant \sum_{k=1}^{n} |\mathbf{V}(t_k)| \frac{\varepsilon}{M(b-a)}\Delta t \leqslant nM\frac{\varepsilon}{M(b-a)} \cdot \frac{b-a}{n} = \varepsilon.$$

That is, $\sum |\mathbf{V}(t_k)| \cos\theta_k \Delta t \to \sum |\mathbf{V}(t_k)| \cos\theta(t_k)\Delta t$ as $n \to \infty$. But the left side approaches $|\mathbf{W}|$, since $R_n \to \mathbf{W}$, and the right is a Riemann sum approximation to $\int_a^b |\mathbf{V}(t)| \cos\theta(t)\,dt$. So $\mathbf{W} = \int_a^b |\mathbf{V}(t)| \cos\theta(t)\,dt$, as claimed.

COROLLARY. *If* $\mathbf{V}(t)$ *is continuous on* $[a,b]$, *then* $\left|\int_a^b \mathbf{V}(t)\,dt\right| \leqslant \int_a^b |\mathbf{V}(t)|\,dt$, *with equality exactly when* $\mathbf{V}(t)$ *has constant polar angle.*

Proof. If $\theta(t)$ is the angle between $\mathbf{V}(t)$ and $\mathbf{W} = \int_a^b \mathbf{V}(t)\,dt$, then since $1 - \cos\theta(t) \geqslant 0$ throughout $[a, b]$, we have $0 \leqslant \int_a^b |\mathbf{V}(t)|\{1 - \cos\theta(t)\}\,dt$, with equality exactly when $\cos\theta(t) \equiv 1$. So, using Lemma 2, $\left|\int_a^b \mathbf{V}(t)\,dt\right| = |\mathbf{W}| = \int_a^b |\mathbf{V}(t)| \cos\theta(t)\,dt \leqslant \int_a^b |\mathbf{V}(t)|\,dt$, with equality just if the angle $\theta(t)$ between $\mathbf{V}(t)$ and $\mathbf{W}$ is zero, which is equivalent to the requirement that $\mathbf{V}(t)$ have constant polar angle.

Proof of the theorem.

$$\left|\int_\gamma f(z)\,dz\right| = \left|\int_a^b f(z(t))z'(t)\,dt\right| \leqslant \int_a^b |f(z(t))||z'(t)|\,dt = \int_a^b |f(z)|\,ds,$$

with equality just if the vector function $f(z(t))z'(t)$ has constant polar angle. But the polar angle of $f(z(t))z'(t)$ is $\arg f(z(t)) + \arg z'(t)$, or $\arg z'(t) - \arg\overline{f(z(t))}$, which we recognize as the angle between the tangent vector to the path and the Pólya vector $\overline{\mathbf{W}}$ at $z(t)$.

Note that the radial vector field for $f(z) = 1/z$ in FIGURE 1 makes a constant angle $\pi/2$ with the tangent vector field on the circle $|z| = 1$, as required in the theorem. And indeed

$$\left|\int_{|z|=1} \frac{1}{z}\,dz\right| = \int_{|z|=1} \left|\frac{1}{z}\right| ds,$$

the common value being 2π. Another example where the constant-angle hypothesis is satisfied is the integral $\int_{\gamma_2} 1/(z^3+1)\,dz$ discussed in Example 2; and the fact that equality holds in the triangle inequality for this integral is immediate from equation (*) of that Example.

In view of the availability of microcomputers* with powerful graphics capabilities, and the increasing number of students familiar with such hardware, it becomes feasible to make class assignments involving use of Pólya vector field pictures in an introductory complex analysis course. Experience with such a geometric model, especially in the study of contour integration, can help eliminate from complex analysis the undesirable connotations of the term "imaginary."

References

[1] Ruel V. Churchill and James W. Brown, Complex Variables and Applications, 4th ed., McGraw-Hill, 1984.

[2] George Pólya and Gordon Latta, Complex Variables, Wiley, 1974.

[3] Bart Braden, Picturing functions of a complex variable, College Math. J., 16 (1985) 63–72.

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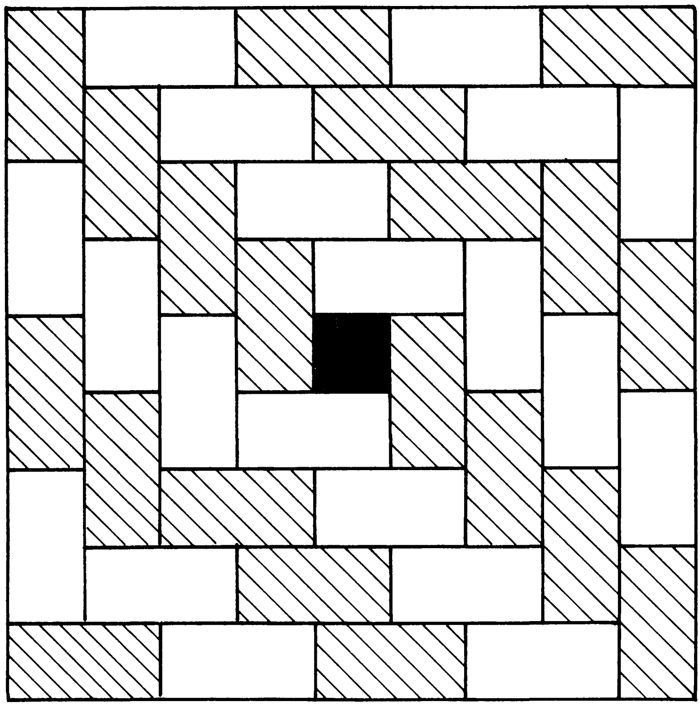
*An advantage of mainframe computers in this connection is their access to mathematical libraries for evaluating functions of a complex variable. A simple program (in FORTRAN 77, for a CalComp plotter) to sketch the Pólya vector field for an arbitrary function $f(z)$ along any specified contour, is available from the author upon request.

Proof without Words: 1 Domino = 2 Squares: Concentric Squares



$$1 + 4 \cdot 2 + 8 \cdot 2 + 12 \cdot 2 + 16 \cdot 2 = 9^2$$

$$1 + 2 \sum_{k=1}^{n} 4k = (2n+1)^2.$$

—SHIRLEY A. WAKIN

University of New Haven

PROBLEMS

LOREN C. LARSON, Editor
BRUCE HANSON, Associate Editor
St. Olaf College

Proposals

Problems especially contributed for this commemorative issue.

To be considered for publication, solutions should be received by May 1, 1988.

1277. *Proposed by Underwood Dudley, DePauw University.*

Determine the next row in the following triangular array of numbers.

```
                              2
                            2   2
                          2   1   2
                        2   0   0   2
                      2   6   5   6   2
                    2   6   4   4   6   2
                  2   6   3   2   3   6   2
                2   6   2   0   0   2   6   2
              2   6   1   9   8   9   1   6   2
            2   6   0   8   6   6   8   0   6   2
          2   6  12   7   4   3   4   7  12   6   2
        2   6  12   6   2   0   0   2   6  12   6   2
      2   6  12   5   0  12  11  12   0   5  12   6   2
    2   6  12   4  14  10   8   8  10  14   4  12   6   2
  2   6  12   3  13   8   5   4   5   8  13   3  12   6   2
2   6  12   2  12   6   2   0   0   2   6  12   2  12   6   2
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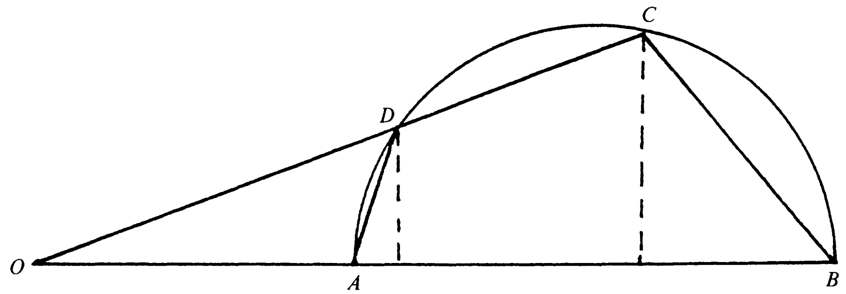
ASSISTANT EDITORS: CLIFTON CORZATT and THEODORE VESSEY, *St. Olaf College. We invite readers to submit problems believed to be new and appealing to students and teachers of advanced undergraduate mathematics. Proposals should be accompanied by solutions, if at all possible, and by any other information that will assist the editors and referees. A problem submitted as a Quickie should have an unexpected, succinct solution. An asterisk (*) next to a problem number indicates that neither the proposer nor the editors supplied a solution.*

Solutions should be written in a style appropriate for Mathematics Magazine. Each solution should begin on a separate sheet containing the solver's name and full address.

Solutions and new proposals should be mailed in duplicate to Loren C. Larson, Department of Mathematics, St. Olaf College, Northfield, MN 55057.

1278. *Proposed by Howard Eves, University of Central Florida, Orlando.*

If O is a given point on the prolongation of diameter BA of a given semicircle, and if ODC is a secant cutting the semicircle in D and C, prove that quadrilateral $ABCD$ has maximum area when the orthogonal projection of DC on AB is equal to the radius of the semicircle.



1279. *Proposed by Abraham P. Hillman, University of New Mexico.*

For nonnegative integers n, let $G(n)$ be the number of integers k such that $\binom{n-k}{k}$ is odd (and $0 \leqslant 2k \leqslant n$).

a. Express $G(2n)$ and $G(2n+1)$ in terms of $G(n)$ and $G(n-1)$.

b. Find $G(2^{1887})$.

(A necessary and sufficient condition using base 2 numerals of m and k for $\binom{m}{k}$ to be odd is known and can be conjectured and proved with the help of

$$(x+1)^{2^a} \equiv x^{2^a} + 1 \pmod{2}.)$$

1280. *Proposed by Donald E. Knuth, Stanford University.*

Prove that

$$\left\lfloor \frac{m^2}{n} \right\rfloor + \sum_{k=0}^{m-1}\left(\left\lfloor \frac{k}{n} \right\rfloor - \left\lfloor \frac{m+k}{n} \right\rfloor\right) = \left\lfloor \frac{\min(m \bmod n, (-m) \bmod n)^2}{n} \right\rfloor$$

for all positive integers m and n. (Here $\lfloor x \rfloor$ denotes the greatest integer $\leqslant x$ and $m \bmod n = m - \lfloor m/n \rfloor n$.)

1281. *Proposed by Murray S. Klamkin, University of Alberta.*

a. Determine the least number of acute dihedral angles in a tetrahedron.

*b. Generalize the result for an n-dimensional simplex. Here a dihedral angle is the supplement of the angle between outward normals to two $(n-1)$-dimensional faces of the simplex.

1282. *Proposed by H. S. M. Coxeter, University of Toronto.*

In Euclidean space, a tetrahedron is called an orthoscheme if its faces consist of four right-angled triangles. How can such a tetrahedron be dissected (by means of two cutting planes) into three pieces each of which is an orthoscheme? Can the three pieces all have the same volume? Are they then congruent?

1283. *Problem due to Leo Moser* (1921–1970), *submitted by William O. J. Moser, McGill University.*

Show that the maximal number of points which can be located on the 4-dimensional unit cube such that all mutual distances are $\geqslant 1$ is 17. Furthermore, the only configuration yielding this number is the center and the 16 vertices.

1284. *Proposed by H.-J. Seiffert, Berlin, West Germany.*

Let $f \in C^2[a, b]$ be strictly positive. Assuming that f and $1/f$ are both convex, prove that

$$\int_a^b \left(\frac{f'(x)}{f(x)}\right)^2 dx \leqslant \frac{1}{b-a} \frac{(f(a)-f(b))^2}{f(a)f(b)}.$$

1285. *Proposed by Paul Erdős and László Lovász, Hungarian School of Sciences.*

Let n be sufficiently large, and let there be given n points in the plane so that every five of them can be covered by two lines. Prove that then all the points can be covered by two lines. Show that this is false for $n = 8$, but holds for $n \geqslant 9$.

1286. *Proposed by Paul C. Rosenbloom, Teachers College, Columbia University.*

Suppose that $F(x)$ is continuous and non-decreasing and g is continuous and non-increasing and positive for $x \geqslant 0$ so that the set E of x such that $F(x+1) - F(x) < g(x)$ is a union, $\bigcup_{n=0}^{\infty} (a_n, b_n)$, of disjoint open intervals, $a_n < b_n \leqslant a_{n+1}$ for all $n \geqslant 0$. Show that $V_F(E)$, the variation of F on E,

$$V_F(E) = \sum_{n=0}^{\infty} (F(b_n) - F(a_n)),$$

satisfies $V_F(E) \leqslant \sum_{n=0}^{\infty} g(n)$.

Quickies

Answers to Quickies are on p. 332.

Q726. *Proposed by C. Kenneth Fan, student, Harvard College.*

Evaluate

$$\lim_{a \to 1^-} \left[(1-a) \sum_{k=0}^{\infty} a^{ke} e^{a^{ke}}\right].$$

Q727. *Proposed by Sidney Kung, Jacksonville University, Florida.*

A circle in $\mathbf{R}^3$ rolls without slipping on the circumference of a fixed circle in such a way that the angle θ between the planes of the circles is constant, $0 < \theta < \pi$. Show that the curve generated by the motion of a point P on the circumference of the rolling circle lies on a sphere.

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PROBLEMS

P, S, Q, *and* C *refer to Proposals, Solutions, Quickies, and Comments, respectively. Page numbers are given in parentheses. Thus,* P1257(39) *refers to proposal number* 1257 *which appears on page* 39.

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Answers

Solutions to the Quickies which appear on p. 330.

A726.

$$\lim_{a\to 1^-}\left[(1-a)\sum_{k=0}^{\infty} a^{ke}e^{a^{ke}}\right]=\lim_{a\to 1^-}\left[\left(\frac{1-a}{1-a^e}\right)\sum_{k=0}^{\infty}(1-a^e)a^{ke}e^{a^{ke}}\right]$$
$$=\left(\lim_{a\to 1^-}\frac{1-a}{1-a^e}\right)\left(\lim_{a\to 1^-}\sum_{k=0}^{\infty}\left(a^{ke}-a^{(k+1)e}\right)e^{a^{ke}}\right)$$
$$=\frac{1}{e}\int_0^1 e^x\,dx=\frac{e-1}{e},$$

since $\{a^{ke}|k=0,1,2,\dots\}$ is a partition of $(0,1]$ and

$$|a^{ke}-a^{(k+1)e}|\leqslant 1-a\to 0 \quad\text{as}\quad a\to 1^-.$$

A727. Let C and $C(t)$ be the fixed circle and the rolling circle at time t, respectively. For a fixed time t, there is a unique sphere which contains C and $C(t)$ on its surface. (The center of this sphere is the intersection of the lines through the centers, and perpendicular to the planes, of C and $C(t)$.) This sphere is independent of t because θ is constant for all t. Thus, all points of $C(t)$, for all t, are on this sphere, and in particular, the curve traced by P is on this sphere.

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